Revision Notes for 6101

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1 Basic Algebra

1.1 Laws of Algebra

1. Given two numbers a and b, then the sum and the product are both numbers. This is written as $a + b$ and $a \times b = a \cdot b = ab$.
2. Addition and multiplication of numbers are *associative*. So given three numbers a, b, c then the following holds: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
3. Addition and multiplication of numbers are *commutative*, the order of addition or multiplication doesn't matter: $a + b = b + a$ and $a \cdot b = b \cdot a$.
4. There are numbers written 0 and 1, which satisfy $a + 0 = 0 + a = a$ and $1 \cdot a = a \cdot 1 = a$
5. There are additive and multiplicative inverses, so there is a number which is written $-a$ such that $a + (-a) = (-a) + a = 0$. Likewise there are numbers denoted a^{-1} such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$.
6. Distributivity of multiplication over addition. Given three numbers a, b, c, the following holds: $a \cdot (b + c) = a \cdot b + a \cdot c$

1.2 Power Laws

1. $x^n \cdot x^m = x^{n+m}$, so $\overbrace{x \cdots x}^{n \text{ times}} \cdot \underbrace{x \cdots x}_{m \text{ times}} = \overbrace{x \cdots x}^{n+m \text{ times}}$
2. $x^n \div x^m = x^{n-m}$
3. If $n = 0$ then we can see as a result of the power law we get:

$$x^{-m} = \frac{1}{x^m}$$

4. $x^0 = 1$
5. $(x^n)^m = x^{nm}$,
6. $x^{1/n} = \sqrt[n]{x}$, this is notation really
7. $x^{m/n} = (x^{1/n})^m$, this applies two previous laws.

1.3 Brackets

$$\begin{aligned}(ax+b)^2 &= a^2x^2+2abx+b^2\\(ax-b)^2 &= a^2x^2-2abx+b^2\\(ax+b)(ax-b) &= a^2x^2-b^2\\(ax+b)(cx+d) &= acx^2+(ad+bc)x+bd\end{aligned}$$

2 Functions

2.1 Basics

A function is a rule which takes one set of numbers (A) to another set of numbers (B). The set A is called the domain and the set B is called the image. Points for which the function is undefined should not be included in the domain.

2.2 Sequences

Sequences are functions whose domain is the natural numbers.

2.3 Composition of Functions

If $f : A \to B$ and $g : B \to C$ then the composition function $f \circ g(x) = g(f(x))$, which means that we take a point $x \in A$ and calculate $f(x) \in B$ and then we take this value and calculate g of it.

2.4 Inverse Functions

The inverse of a function $f(x)$ is denoted as $f^{-1}(x)$ and defined as:

$$f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$$

3 Linear Equations

3.1 Basics

A *linear equation* is an equation of the form:

$$y = mx + c$$

Where m is the gradient and c is the y-intercept. Given two points (x_1, y_1) and (x_2, y_2), the gradient, m of the line will be given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The y-intercept is given by:

$$c = y_1 - \frac{y_2 - y_1}{x_2 - x_1}x_1$$

The gradient of the normal to a line with gradient m is given by $-1/m$.

3.2 Intersection of Lines

Given two lines:

$$\begin{aligned} ay + bx &= c \\ dy + ex &= f \end{aligned}$$

The point of intersection is the point (x, y) which satisfies both the equations (examples given in class)

3.3 Shortest Distance From a Point to a Line

The shortest distance will be on a normal from the line. Suppose the point in (a, b) and the line is given by $y = mx + c$, do the following steps:

1. Find the gradient(n) of the normal line (this is $n = -1/m$)
2. Compute the equation of the normal line going through the point (a, b), write the equation of the normal line as $y = nx + d$ and set $x = a$ and $y = b$ to find d.
3. Compute the point of intersection of the normal line and the original line.
4. Calculate the distance between these two points via Pythagoras's theorem.

4 Quadratic Equations

4.1 Basics

A quadratic equation is of the form:

$$y = ax^2 + bx + c$$

The completed square form if the above equation is given by:

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

The *roots* of the quadratic are given be setting $y = 0$ in the above equations, there are two roots given by:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1)$$

A *root*,α of any polynomial equation $f(x) = a_1x^n + a_2x^{n-1} + \cdot + a_{n-1}x + a_n$ is given by $f(\alpha) = 0$.[1]

[1] A good place to start to find roots of polynomials is start with $x = 1$

4.2 Maxima and Minima

A quadratic $y = ax^2 + bx + c$ will have a minimum if $a > 0$ and a maximum if $a < 0$. The x-co-ordinate at which this happens is:

$$x = -\frac{b}{2a}$$

The y-value at which this happens is:

$$y = -\frac{b^2 - 4ac}{4a}$$

4.3 Odd and Even Functions

An *odd* function is where $f(-x) = -f(x)$ and an *even* function is where $f(-x) = f(x)$.

4.4 Circles

The equation of a circle of radius r with centre at (a, b) is given by:

$$(y - b)^2 + (x - a)^2 = r^2$$

5 Limits

5.1 Algebra of Limits

let $a_n \to a$ and $b_n \to b$, then

$$\begin{aligned} a_n + b_n &\to a + b \\ a_n b_n &\to ab \\ \frac{a_n}{b_n} &\to \frac{a}{b} \quad b \neq 0 \end{aligned}$$

5.2 Algebra of Functions

Let $f(x) \to f$ as $x \to a$ and $g(x) \to g$ as $x \to a$, then as $x \to a$:

$$\begin{aligned} f(x) + g(x) &\to f + g \\ f(x)g(x) &\to fg \\ \frac{f(x)}{g(x)} &\to \frac{f}{g} \quad g \neq 0 \end{aligned}$$

5.3 Limit as $x \to \infty$

Set $y = 1/x$, rearrange and compute the limit as $y \to 0$

5.4 One-Sides Limits

The *lower limit* is defined as:

$$\lim_{x \nearrow a} f(x) = \lim_{x \to 0} f(x-h)$$

The *upper limit* is defined as:

$$\lim_{x \searrow a} f(x) = \lim_{x \to 0} f(x+h)$$

5.5 Continuity

If:

$$\lim_{x \nearrow a} f(x) = \lim_{x \searrow a} f(x) = f(a)$$

then the function is continuous.

6 Differentiation

6.1 Basics

A function $f(x)$ is differentiable if the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Some well known derivatives:

$$\begin{aligned}
\frac{d}{dx}(ax+b)^n &= an(ax+b)^{n-1} \\
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \frac{1}{\cos^2} \\
\frac{d}{dx}e^x &= e^x \\
\frac{d}{dx}\ln x &= \frac{1}{x}
\end{aligned}$$

Rules for differentiation:

$$\begin{aligned}
\frac{d}{dx} f(x)g(x) &= f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx} \\
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{g(x)^2} \\
\frac{d}{dx} f \circ g &= f'(g(x))g'(x)
\end{aligned}$$

6.2 L'Hopital's Rule

If:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

6.3 MacLaurin Series

If $f(x)$ can be differentiated lots of times that it may be expressed as a series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

6.4 Stationary Points and Extrema

$f(x)$ has a *stationary point* at $x = a$ when $f'(a) = 0$. If

$$\left.\frac{d^2 f}{dx^2}\right|_{x=a} < 0$$

then the point is a *maximum* and if:

$$\left.\frac{d^2 f}{dx^2}\right|_{x=a} > 0$$

The point is a minimum.

7 Trigonometry

7.1 Basics

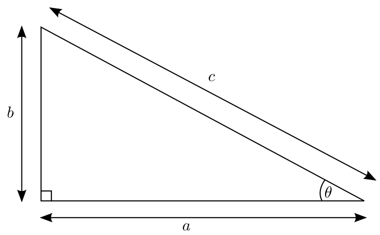

Figure 1: A Right-Angled Triangle

The trigonometric functions are defined as:

$$\sin\theta = \frac{b}{c}, \quad \cos\theta = \frac{a}{c}, \quad \tan\theta = \frac{b}{a}$$

7.2 Properties of Trig Functions

The basic properties are:

$$\sin(\theta + 2\pi) = \sin\theta, \quad \cos(\theta + 2\pi) = \cos\theta, \quad \tan(\theta + \pi) = \tan\theta$$

Some basic values of the trig functions are:

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}, \quad \tan\frac{\pi}{3} = \sqrt{3}$$

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \tan\frac{\pi}{4} = 1$$

8 Double Angle Formulae

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \sin B\cos A \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A\tan B} \\
\tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

8.1 Trig Identities

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &\equiv 1 \\
1 + \tan^2\theta &\equiv \sec^2\theta \\
1 + \cot^2\theta &\equiv \mathrm{cosec}^2\theta
\end{aligned}$$

9 Exponentials and Logarithms

9.1 Basics

The definition of e is:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

The exponential function is defined as: The definition of e is:

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

The logarithm is defined to be the inverse of the exponential function, so:

$$\begin{aligned}
\ln(e^x) &= x \\
e^{\ln(x)} &= x
\end{aligned}$$

The log has the following properties:

$$\begin{aligned} \ln(ab) &= \ln a + \ln b \\ \ln x^a &= a \ln x \end{aligned}$$

9.2 General Exponentials and Logarithms

Use the following definitions:

$$\begin{aligned} a^x &= e^{x \ln a} \\ \log_a x &= \frac{\ln x}{\ln a} \end{aligned}$$

10 Complex Numbers

10.1 Basics

The imaginary number i is defined as $i^2 = -1$, a complex number, z is made up of a real number, x and an imaginary number yi and is written $z = x + yi$, The complex conjugate of a complex number is written as $\bar{z} = x - yi$, the modulus of a complex number is $|z|^2 = z\bar{z} = x^2 + y^2 = r$. The argument of a complex number is defined as:

$$Arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The complex number can be alternatively written $z = r(\cos\theta + i\sin\theta)$

11 Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

11.1 DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$