Introduction to Integration

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Abstract

This set of notes provides an introduction to integration, what it is and some of the techniques that are used. The numerical evaluation of the integral via the trapezium rule. The level of material will cover up to A-level.

1 Finding the Area Under a Curve

1.1 Basic Theory

Suppose we have a curve $y = f(x)$ and we want to find the area under the graph from a point $x = a$ to a point $x = b$. One way that we can possibly

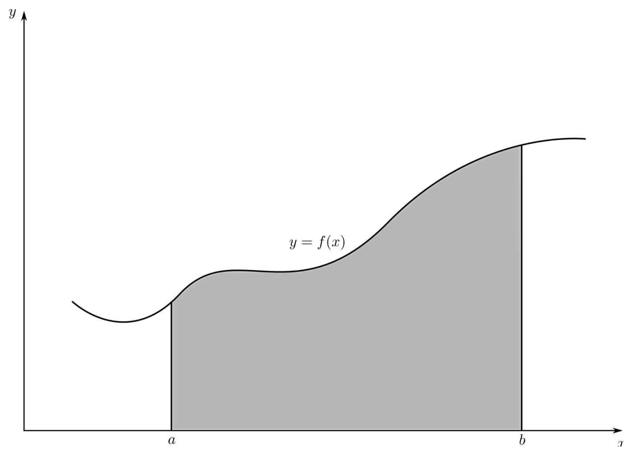

Figure 1: Finding the area under the graph

find the area is to split the area up into strips and then the area of the strips can be calculated. The thinner the strip then the more accurate the answer for the area. Say we split up the interval $[a, b]$ into n pieces each of length $(b-a)/n$, the points of the interval are

$$x_i = a + \left(\frac{b-a}{n}\right) i \tag{1}$$

The set of points $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ is called a *partition* of the interval $[a, b]$. To compute the area choose strips that lie completely within the area: Mathematically we are choosing the minimum

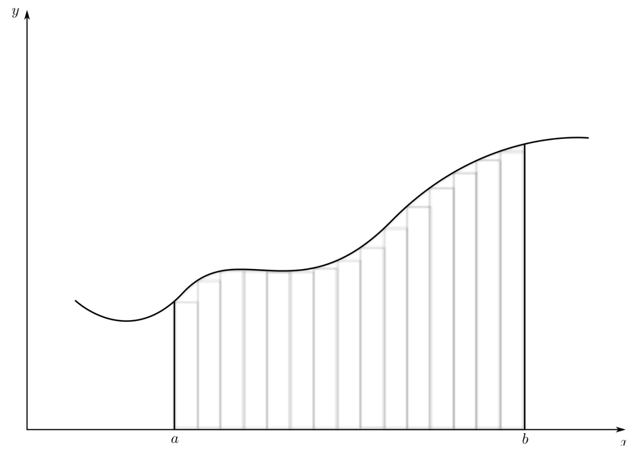

Figure 2: Underestimating the area

of $f(x_i)$ and $f(x_{i+1})$ for the strip starting at the point x_i. Hopefully when a larger and larger values of n is chosen then the area of the strips will converge to the area under the graph. The area under the curve can then be estimated as the following:

$$A_L(n) = \sum_{i=0}^{n-1} \min(f(x_i), f(x_{i+1}))(x_{i+1} - x_i) = \frac{b-a}{n} \sum_{i=0}^{n-1} \min(f(x_i), f(x_{i+1})) \tag{2}$$

This will be called the *lower sum* for the area. Note that the term $x_{i+1} - x_i$ is used for the interval $(b-a)/n$ because in general the partition can be uneven in the widths of the strips $x_{i+1} - x_i$, we have made then even for convenience only. Likewise we can overestimate the area under the curve by choosing the maximum of $f(x_i)$ and $f(x_{i+1})$ for a strip starting at x_i The area under the

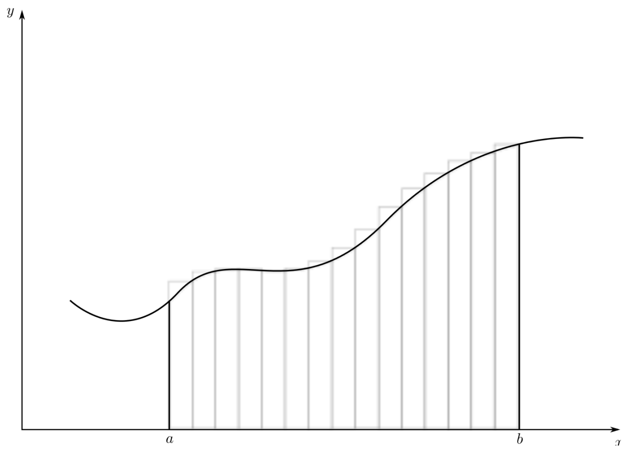

Figure 3: Overestimating the area

curve can then be estimated as the following:

$$A_U(n) = \sum_{i=0}^{n-1} \max(f(x_i), f(x_{i+1}))(x_{i+1} - x_i) = \frac{b-a}{n} \sum_{i=0}^{n-1} \max(f(x_i), f(x_{i+1})) \tag{3}$$

This will be called the *upper sum* for the area. The approximations will be valid if we can find an $n > N$ for which $|A_U(n) - A_L(n)| < \varepsilon$, this is called the *Riemann criterion*. The Riemann criterion basically states that if the difference of the upper and lower limits is vanishingly small as n gets larger and larger, then the approximations we made are valid for computing the area under the curve. Once we know that the Riemann criterion holds then we can take the limit as $n \to \infty$ in either the upper or lower limits to obtain the area.

Example. Take the equation $y = x$ and we will compute the area under the curve from the point $x = 0$ to $x = a$. The interval $[0, a]$ would be split up into n different points, each of length a/n, the points in the partition are ai/n. Then computing the lower limit of the area shows that:

$$A_L(n) = \frac{a}{n} \sum_{i=0}^{n-1} \frac{ai}{n} = \frac{a^2}{n^2} \sum_{i=0}^{n-1} i \tag{4}$$

Now:

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

Then:

$$A_L(n) = \frac{a^2}{2}\left(1 - \frac{1}{n}\right) \tag{5}$$

Now to compute The upper sum of the area, we have:

$$A_U(n) = \frac{a}{n}\sum_{i=1}^{n}\frac{ai}{n} = \frac{a^2}{2}\sum_{i=1}^{n} i = \frac{a^2}{2}\left(1 + \frac{1}{n}\right) \tag{6}$$

Now geometrically $A_U(n) > A_L(n)$ and so $|A_U(n) - A_L(n)| = A_U(n) - A_L(n)$, so computing this condition shows that:

$$A_U(n) - A_L(n) = \frac{a^2}{2}\left(1 + \frac{1}{n}\right) - \frac{a^2}{2}\left(1 - \frac{1}{n}\right) = \frac{a^2}{n}$$

So taking $N > a^2/\varepsilon$ will ensure that the the Riemann criterion will hold and the approximations for the area under the curve is valid and we may take the limit as $n \to \infty$ to obtain the *exact* area under the curve to obtain the area to be $a^2/2$ which is geometrically intuitive because we computed the area of a right angled triangle which is just half the area of a square with length of sides a.

Example. Take the function $y = x^2$ and as before we will compute the area under the graph from $x = 0$ to $x = a$. The partition of the interval $[0, a]$ will be divided equally into n pieces, each of length a/n and the points of the partition will be ai/n. Computing the lower sum shows that:

$$A_L(n) = \frac{a}{n}\sum_{i=0}^{n-1}\left(\frac{ai}{n}\right)^2 = \frac{a^3}{n^3}\sum_{i=0}^{n-1} i^2 \tag{7}$$

Now using the fact that:

$$\sum_{i=0}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$$

shows that:

$$A_L(n) = \frac{a^3}{n^3}\frac{n(n-1)(2n-1)}{6} = \frac{a^3}{6}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right) \tag{8}$$

Computing the upper limit shows that:

$$A_U(n) = \frac{a}{n}\sum_{i=1}^{n}\left(\frac{ai}{n}\right)^2 = \frac{a^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \tag{9}$$

Examining the Riemann criterion, we have:

$$A_U(n) - A_L(n) = \frac{a^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - \frac{a^3}{6}\left(2 - \frac{3}{n} + \frac{1}{n^2}\right) = \frac{a^3}{n}. \tag{10}$$

So this can be made less that any given ε by taking $N > a^3/\varepsilon$. So the Riemann criterion is satisfied and we may take the limit as $n \to \infty$ to obtain the area as $a^3/3$.

Definition The *upper integral* from a point $x = a$ and $x = b$ of a function $y = f(x)$ is denoted as:

$$\overline{\int_a^b} f(x)dx \tag{11}$$

and is defined as:

$$\overline{\int_a^b} f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \max(f(x_i), f(x_{i+1}))(x_{i+1} - x_i). \tag{12}$$

The *lower integral* from a point $x = a$ and $x = b$ of a function $y = f(x)$ is denoted as:

$$\underline{\int_a^b} f(x)dx \tag{13}$$

and is defined as:

$$\underline{\int_a^b} f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \min(f(x_i), f(x_{i+1}))(x_{i+1} - x_i). \tag{14}$$

If the Riemann criterion holds then the upper integral is the same as lower integral and we simply call it the *integral* and we write it as:

$$\int_a^b f(x)dx \tag{15}$$

Geometrically the integral represents the area under the curve from the points $x = a$ to the point $x = b$. The point a is called the *lower limit* and the point b is called the *upper limit*. Functions which can be integrated are called *integrable*. The function $f(x)$ inside the integral is called the *integrand*. It should be noted that x in the integral is what is known as a *dummy variable* and any variable can be used. It is good practice to use a different letter to represent the dummy variable than what appears in the limits.

Integration is the act of performing an integral and along with differentiation makes up a subject called *calculus*. Integration is sometimes called *integral calculus*

1.2 Basic Properties

There are three useful properties of the integral:

$$\int_a^a f(x)dx = 0 \tag{16}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad a < c < b \tag{17}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{18}$$

It is easy to see why the first of these properties holds. The geometric interpretation of an integral is the area under the graph, so the first of the properties ask about the area under the graph from $x = a$ to $x = a$ and this is 0. For the second statement, known as the *domain splitting property*, consider the following illustration: From the above diagram it can be seen

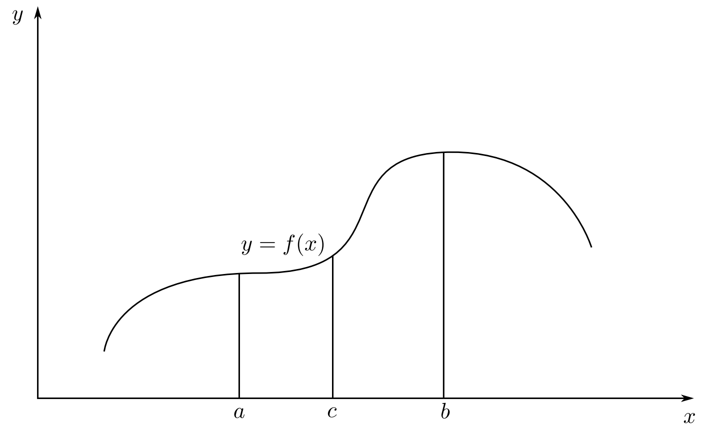

Figure 4: Domain splitting

quite intuitively that the integrals from a to c is just the area under the graph from a to c and the integral from c to b is just the area under the graph from c to b. However when you look at the whole, this is just the area under the graph of $f(x)$ from a to b and it is clear why we can split up the integral in the way we did. This property is very useful on occasion. The last property follows from the first two because:

$$\begin{aligned} 0 &= \int_a^a f(x)dx \\ &= \int_a^b f(x)dx + \int_b^a f(x)dx, \end{aligned}$$

and so:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Although the above properties may at first sight seem somewhat useless, they are however a very useful set of properties.

2 Fundamental Theorem of Calculus

2.1 The Main Result

The processes of differentiation and integration are related to each other in a very close way in that one in the reverse of the other. To see this define let $f(x)$ be some function which can be integrable and define $F(x)$ by the equation:

$$F(x) = \int_a^x f(u)dx \tag{19}$$

The geometric picture for the fundamental theorem of calculus is given by: The quotient for the derivative is given by:

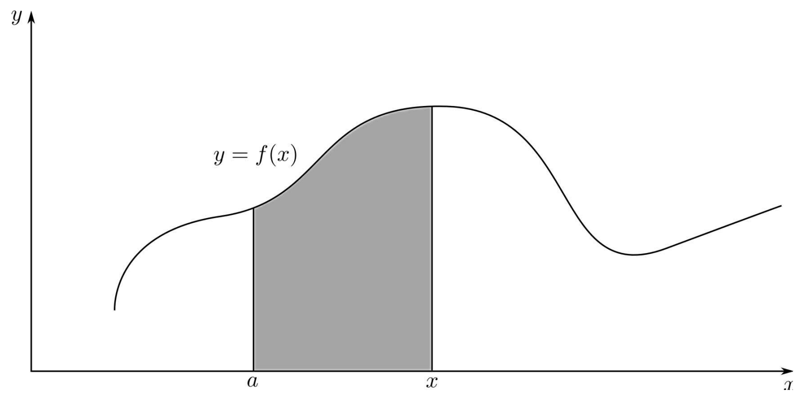

Figure 5: Fundamental Theorem of Calculus

$$\begin{aligned} \frac{F(y)-F(x)}{y-x} &= \frac{\int_a^y f(u)du - \int_a^x f(u)du}{y-x} \\ &= \frac{\int_a^y f(u)du + \int_x^a f(u)du}{y-x} \\ &= \frac{1}{y-x}\int_x^y f(u)du \end{aligned}$$

Now the task is to compute the integral. The way to compute the integral

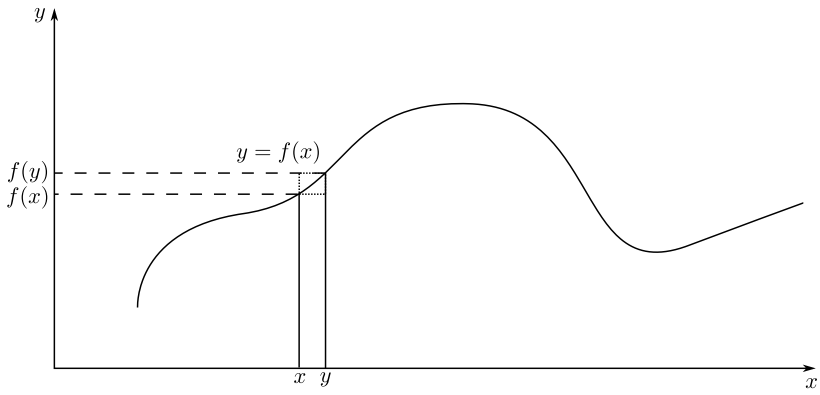

Figure 6: Computing the integral

is to go back to the definition, we find the upper and lower sums for the integral. As we are seeking to differentiate, which means to take the limit, we can suppose that the points x and y are close together the we can suppose that the integral can be approximated as shown in figure 6. It could be the case that the function is decreasing on the interval $[x, y]$, in which case the roles of $f(x)$ and $f(y)$ swap. It is always possible to arrange it so the function in question is always either increasing or decreasing on the interval $[x, y]$. Suppose there is a stationary point in the interval $[a, x]$ at a point c, where $a < c < x$ then the property of domain splitting can be used. Split up the domain as:

$$\int_a^x f(u)du = \int_a^c f(u)du + \int_c^x f(u)du$$

So the function will either be increasing or decreasing on the interval $[x, y]$. The lower sum is simply given by:

$$A_L = f(x)(y - x)$$

and the upper sum is given by:

$$A_U = f(y)(y - x)$$

So the integral is bounded between:

$$\frac{A_L}{y - x} \leqslant \frac{1}{y - x} \int_x^y f(u)du \leqslant \frac{A_U}{y - x},$$

which shows that:

$$f(x) \leqslant \frac{1}{y - x} \int_x^y f(u)du \leqslant f(y). \tag{20}$$

Which also means that:

$$f(x) \leqslant \frac{F(y) - F(x)}{y - x} \leqslant f(y). \tag{21}$$

So taking the limit as $y \to x$, shows that

$$\boxed{F'(x) = f(x).} \tag{22}$$

Any function $F(x)$ which satisfies (22) is called an *antiderivative* or *primitive* of $f(x)$. All antiderivatives of a particular function differ by a constant only, to show this let $F_1(x)$ and $F_2(x)$ be antiderivatives for a function $f(x)$, so $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$ and suppose the difference of the antiderivatives is another function $g(x)$, so:

$$F_1(x) - F_2(x) = g(x) \tag{23}$$

Differentiation of equation (23) shows that:

$$g'(x) = 0$$

and so $g(x)$ is constant. Suppose that $F(x)$ and $G(x)$ are antiderivatives for $f(x)$, so $F(x) = G(x) + C$ and $F(x)$ is defined as:

$$F(x) = \int_a^x f(x)dx$$

Then evaluation the equation for the antiderivatives at $x = a$ shows that:

$$\int_a^a f(x)dx = 0 = G(a) + C \tag{24}$$

and so $C = -G(a)$ evaluating the equation at $x = b$ shows that $F(b) = G(b) + C = G(b) - G(a)$, and so:

$$\boxed{\int_a^b f(x)dx = G(b) - G(a) = [G(x)]_a^b} \tag{25}$$

Define $h(x)$ as an antiderivative of $f'(x)$:

$$h(x) = \int_a^x f'(x)dx,$$

so $h(x)$ and $f(x)$ will differ by a constant (as both $h(x)$ and $f(x)$ are antiderivatives for $f'(x)$), $h(x) - f(x) = C$, evaluating this at $x = a$ shows that

$f(a) = -C$ and evaluating it at $x = b$ shows that $h(b) - f(a) = C = -f(a)$ and so $h(b) = f(b) - f(a)$, which implies:

$$\boxed{\int_a^b f'(x)dx = f(b) - f(a) = [f(x)]_a^b} \tag{26}$$

The square brackets is just notation for taking the difference. So what we have shown here is that differentiation and integration are truly inverse processes of each other, this very important result is known as the ***fundamental theorem of calculus***. The fundamental theorem of calculus allows is to immediately write down a whole load of integrals without having to resort to computing the upper and lower sums. It is common practice to call integrals of the form:

$$F(x) = \int_a^b f(x)dx \tag{27}$$

indefinite integrals or *improper integrals* and just write them as:

$$F(x) = \int f(x)dx + C, \tag{28}$$

where C is an arbitrary constant (remember antiderivatives of the same function differ by a constant). As an example examine the function:

$$f(x) = \frac{1}{n+1}x^{n+1}$$

Differentiating this shows that $f'(x) = x^n$, and so we can write:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \tag{29}$$

2.2 Linearity of the Integral

Given two functions $f(x)$ and $g(x)$ with primitives $F(x)$ and $G(x)$ respectively and let α and β be two numbers, then the following result holds for the integral:

$$\boxed{\int_a^b \alpha f(x) + \beta g(x)dx = \alpha \int_a^b f(x) + \beta \int_a^b g(x)dx} \tag{30}$$

This very important property is called *linearity.* to show this define $H(x) = \alpha F(x) + \beta G(x)$, differentiate this to obtain:

$$H'(x) = \alpha f(x) + \beta g(x)$$

Integrating this result from a to b shows that:

$$\int_a^b \alpha f(x) + \beta g(x) dx = \int_a^b H'(x) dx \tag{31}$$

The RHS of (31) can be evaluated as follows:

$$\begin{aligned}
\int_a^b H'(x)dx &= [H(x)]_a^b \\
&= H(b) - H(a) \\
&= \alpha F(b) + \beta G(b) - \alpha F(a) - \beta G(a) \\
&= \alpha(F(b) - F(b)) + \beta(G(b) - G(a)) \\
&= \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx
\end{aligned}$$

Combining these results give the result sought after. Although this doesn't seem all that useful at first sight, it is indeed one of the most important properties that the integral has, for example we can evaluate the following integral:

$$\int x^2 + 4x^3 dx = \int x^2 dx + 4 \int x^3 dx = \frac{x^3}{3} + x^4 + C$$

There are numerous examples of when linearity is useful in calculus.

2.3 Negative Area

Examine the integral of $f(x) = x^3$ from $x = -a$ to $x = 0$, so:

$$\int_{-a}^0 x^3 dx = \left[\frac{x^4}{4}\right]_{-a}^0 = \frac{0^4}{4} - \frac{(-a)^4}{4} = -\frac{a^4}{4}$$

This gives a negative result, which is odd as previously we have given the geometrical interpretation as the area below the graph and the $y = 0$ axis which is positive. However when we pot the graph of $y = x^3$ we notice that the the values of $f(x) = x^3$ are negative. As $f(x) = x^3$ is an odd function, it is possible to find the area under the graph between $x = 0$ and $x = a$ as:

$$\int_0^a x^3 dx = \left[\frac{x^4}{4}\right]_0^a = \frac{a^4}{4} - \frac{0^4}{4} = \frac{a^4}{4}$$

So from this calculation we can see that this represents the area under the graph as it gives a positive result. So the geometrical interpretation of the

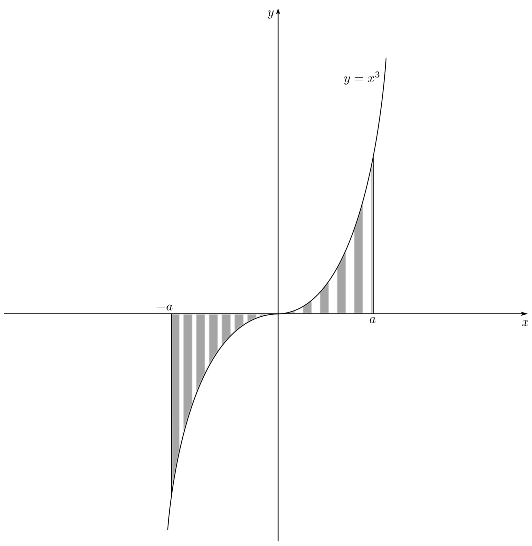

Figure 7: Negative Area

negative result is the area *above* the graph to the y-axis. So we can write down the general rule that if $f(x)$ is an *odd* function:

$$\boxed{\int_{-a}^{a} f(x)dx = 0} \tag{32}$$

and if $g(x)$ is an *even* function then:

$$\boxed{\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx} \tag{33}$$

3 Some Standard Integrals

3.1 Polynomials

This section we show how the standard results for integration may be obtained, one was covered earlier:

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C,}$$

this was obtained by spotting an antiderivative of x^n and this is how some of the integrals are found.

3.2 Trigonometric Functions

The integral of trigonometric functions are found this way, note that:

$$\frac{d}{dx}\left(\frac{1}{a}\sin ax\right) = \cos ax \quad \frac{d}{dx}\left(-\frac{1}{a}\cos ax\right) = \sin ax$$

and so we can say:

$$\boxed{\int \sin ax dx = -\frac{1}{a}\cos ax + C \quad \int \cos ax dx = \frac{1}{a}\sin ax + C.} \qquad (34)$$

Note that for $\tan x$, we have:

$$\frac{d}{dx}\tan x = \sec^2 x \quad \Rightarrow \quad \boxed{\int \sec^2 x dx = \tan x + C} \qquad (35)$$

3.2.1 Integrating $\sin^n x \cos x$ and $\cos^n x \sin x$

Differentiating $f(x) = \sin^{n+1} x$ and $f(x) = \cos^{n+1} x$ yields:

$$\frac{d}{dx}\left(\sin^{n+1} x\right) = (n+1)\sin^n x \cos x, \quad \frac{d}{dx}\left(\cos^{n+1} x\right) = -(n+1)\cos^n x \sin x$$

and therefore:

$$\boxed{\int \sin^n x \cos x dx = \frac{1}{n+1}\sin^{n+1} x + C},$$

$$\boxed{\int \cos^n x \sin x dx = -\frac{1}{n+1}\cos^{n+1} x + C}$$

The above results are useful for computing the integrals of odd powers of $\sin x$ and $\cos x$. As an example of the method let's integrate $f(x) = \sin^3 x$. now $\sin^3 x = \sin^2 x \cdot \sin x$ and using the trigonometric identity $\sin^2 x + \cos^2 x \equiv 1$ in the form of $\sin^2 x = 1 - \cos^2 x$ shows that $\sin^3 x = (1 - \cos^2 x)\sin x$ and so:

$$\begin{aligned}
\int \sin^3 x dx &= \int (1 - \cos^2 x)\sin x dx \\
&= \int \sin x dx - \int \cos^2 x \sin x dx \\
&= -\cos x + \frac{1}{3}\cos^3 x + C
\end{aligned}$$

The same idea can be used for integrating odd powers of $\cos x$.

3.2.2 Integrating Even Powers of $\cos x$ and $\sin x$

There is no general formula for this but there is a general method, the method us based upon the expression for $\cos 2x$, as:

$$\cos 2x = 2\cos^2 x - 1$$

So, integrate $f(x) = \cos^4 x$. Note $\cos^4 x = (\cos^2 x)^2 = ((1 + \cos 2x)/2)^2$, expanding this out completely shows that:

$$\cos^4 x = \frac{1 + 2\cos 2x + \cos^2 2x}{4}$$

Using $\cos^2 2x = (1 + \cos 4x)/2$ shows that:

$$\cos^4 x = \frac{1}{4} + \frac{\cos 2x}{2} + \frac{1 + \cos 4x}{8}$$

The result can now be integrated using the standard rules given before.
For even powers of $\sin x$, turn $\sin^2 x$ into $1 - \cos^2 x$ any apply the technique above.

3.2.3 Integration of $\tan^n x$

First off examine the derivative of $\tan^n x$ which is:

$$\frac{d}{dx}(\tan^n x) = n\tan^{n-1} x \sec^2 x,$$

so we can conclude that:

$$\boxed{\int \tan^n x \sec^2 x dx = \frac{1}{n+1}\tan^{n+1} x + C.}$$

Use $\tan^n x = \tan^2 x \tan^{n-2} x = \tan^{n-2} x(\sec^2 x - 1)$ and we are lead to the relation:

$$\boxed{\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx} \qquad (36)$$

Equations of type (36) are called *recurrence relations*. When $n = 2$ we have:

$$\int \tan^2 x dx = \int \sec^2 x - 1 dx = \tan x - x + C$$

This completes all the integration for $\tan^n x$.

3.3 Exponentials and Logarithms

A basic result is:

$$\frac{d}{dx}\left(\frac{1}{a}e^{ax}\right) = e^{ax}$$

so:

$$\boxed{\int e^{ax}dx = \frac{1}{a}e^{ax} + C.} \tag{37}$$

Likewise:

$$\frac{d}{dx}\left(\frac{1}{\ln a}a^{x}\right) = a^{x}$$

and so:

$$\boxed{\int a^{x}dx = \frac{1}{\ln a}a^{x} + C} \tag{38}$$

We know that if $f(x) = \ln x$ then $f'(x) = x^{-1}$ when $x > 0$ and so we have:

$$\int \frac{1}{x}dx = \ln x + C, \quad x > 0 \tag{39}$$

However for $x < 0$, we have the following picture: As was discussed earlier,

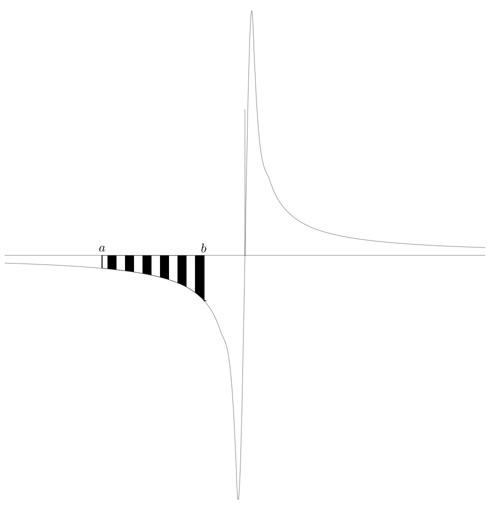

Figure 8: Integral of $\ln x$ for $x < 0$

the value of the integral:

$$\int_a^b \frac{1}{x} dx$$

should have a value as the area above the curve us well defined geometrically, to get a way around this multiple both denominator and numerator by -1 to get:

$$\int_a^b \frac{-1}{-x} dx = \int_a^b \frac{1}{-x} d(-x) = [\ln(-x)]_a^b,$$

for values of $x < 0$. So we can see that in general that:

$$\boxed{\int \frac{1}{x} dx = \ln|x| + C} \tag{40}$$

Generalising this result slightly, note that if $f(x) = \ln(f(x))$, then:

$$f'(x) = \frac{f'(x)}{f(x)}, \tag{41}$$

and so:

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C.} \tag{42}$$

Care has to be taken in correctly recognising the intergrand in these cases, for example integrating $f(x) = x/(1 + x^2)$

$$\int \frac{x}{1 + x^2} dx = \frac{1}{2} \int \frac{2x}{1 + x^2} dx,$$

and this can be seen to be in the correct form for $f(x) = 1 + x^2$ and so:

$$\int \frac{x}{1 + x^2} = \frac{1}{2} \ln|1 + x^2| + C.$$

In this case the modulus signs can be dropped as $1 + x^2 > 0$ for all x and we simply write:

$$\int \frac{x}{1 + x^2} dx = \frac{1}{2} \ln(1 + x^2) + C$$

Example, integrate the function $f(x) = x/(1 + x)$. Some manipulation has to be done in order to get this into a suitable form.

$$\begin{aligned} \int \frac{x}{1 + x} dx &= \int \frac{x + 1 - 1}{1 + x} \quad \text{This is the "trick" of the integral} \\ &= \int \frac{1 + x}{1 + x} - \frac{1}{1 + x} dx \\ &= \int 1 - \frac{1}{1 + x} dx \\ &= x - \ln|1 + x| + C. \end{aligned}$$

The trick here was to add and subtract 1 in the numerator. Another application of this in integrating $\tan x$, note that $\cos' x = -\sin x$ and so:

$$\int \tan x dx = -\ln|\cos x| + C.$$

4 Integration by Parts

The product rule for differentiation is given by:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \tag{43}$$

Now integrating (43) from a to b shows that:

$$\int_a^b \frac{d}{dx}(uv)dx = \int_a^b v\frac{du}{dx}dx + \int_a^b u\frac{dv}{dx}dx \tag{44}$$

Using the fundamental theorem of calculus on the LHS of (44) shows that:

$$\int_a^b \frac{d}{dx}(uv)dx = [u(x)v(x)]_a^b \tag{45}$$

Inserting this into (44) shows that:

$$[u(x)v(x)]_a^b = \int_a^b v\frac{du}{dx}dx + \int_a^b u\frac{dv}{dx}dx. \tag{46}$$

Taking the first term in the RHS over to the LHS gives the equation:

$$\boxed{\int_a^b u\frac{dv}{dx}dx = [u(x)v(x)]_a^b - \int_a^b v\frac{du}{dx}dx} \tag{47}$$

This is the expression for *integration by parts*. The indefinite integral version of this expression is:

$$\boxed{\int u\frac{dv}{dx}dx = u(x)v(x) - \int v\frac{du}{dx}dx} \tag{48}$$

Example Find the indefinite integral of $f(x) = xe^x$, so we compute:

$$\int xe^x dx$$

Use $u = x$ and $dv/dx = x^x$ then:

$$\frac{du}{dx} = 1, \quad v = e^x$$

Inserting these into the integration by parts formula shows that:

$$\int xe^x dx = xe^x - \int 1 \cdot e^x dx = xe^x + e^x + C$$

As can be seen, some care is needed in correctly choosing the values for u and v. If we had chosen $dv/dx = x$ and $u = x$ then we would be left with the integral:

$$\int xe^x dx = \frac{x^2 e^x}{2} - \frac{1}{2}\int x^2 e^x dx$$

which is a more complicated than the original integral.
Example. Find the indefinite integral of $f(x) = e^x \sin x$, so we compute:

$$\int e^x \sin x dx$$

For this let $u(x) = e^x$ and $dv/dx = \sin x$, then:

$$\frac{du}{dx} = e^x, \quad v(x) = -\cos x$$

Then using the integration by parts equation shows that:

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx \tag{49}$$

So now we have to find:

$$\int e^x \cos x dx$$

So choosing $u = e^x$ and $dv/dx = \cos x$, and so:

$$\frac{du}{dx} = e^x, \quad v = \sin x$$

The integration by part formula says that:

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx \tag{50}$$

Inserting (50) into (49) shows that:

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx.$$

Upon re-arranging

$$\int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Example Integrate $f(x) = \ln x$, so we compute:

$$\int \ln x dx = \int 1 \cdot \ln x dx$$

Using $u = \ln x$ and $dv/dx = 1$, so:

$$\frac{du}{dx} = \frac{1}{x}, \quad v = x$$

so the integral becomes:

$$\int \ln x dx = x \ln x - \int \frac{1}{x} x dx = x \ln x - \int 1 dx = x \ln x - x + C$$

5 Integration by Substitution

5.1 Basic Idea

Define:

$$F(x) = \int_c^x f(u) du.$$

The next step is to examine $(F(g(x)))'$ which by the chain rule is:

$$(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x) \tag{51}$$

Then by the fundamental theorem of calculus:

$$\int_a^b (F(g(x)))' dx = F(g(b)) - F(g(a)). \tag{52}$$

Now:

$$F(g(b)) - F(g(a)) = \int_c^{g(b)} f(u) du - \int_c^{g(a)} f(u) du = \int_{g(a)}^{g(b)} f(u) du \tag{53}$$

Inserting (53) and (51) into (52) shows that:

$$\boxed{\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du} \tag{54}$$

The way that integration by substitution usually works is that we are given an integral usually[1] in the form of the RHS of (54) and our task to to come up with a substitution $u = g(x)$ that will simplify the integration.

[1]Sometimes it is in the form of the LHS, an example is given of this

5.2 Examples

Example. Compute the following integral:

$$\int_0^x \frac{du}{1+u^2}$$

Define $u(v) = \tan v$, then $u'(v) = \sec^2 v$ The limits transform as $u = 0 \Rightarrow v = 0$ and likewise $u = x \Rightarrow v = \tan^{-1} x$. Inserting this into (54) shows that:

$$\int_0^x \frac{du}{1+u^2} = \int_0^{\tan^{-1} x} \frac{\sec^2 v}{1+\tan^2 v} dv = \int_0^{\tan^{-1} x} 1 dv = [v]_0^{\tan^{-1} x} = \tan^{-1} x$$

Example. Evaluate:

$$\int_0^{\sqrt{3}} \frac{2x}{\sqrt{1+x^2}} dx$$

Use $g(x) = 1 + x^2$, then $g'(x) = 2x$, the limits transform as $x = 0 \Rightarrow g = 0$ and $x = \sqrt{3} \Rightarrow g = 4$, then inserting these values in (54) shows that:

$$\int_0^{\sqrt{3}} \frac{2x}{\sqrt{1+x^2}} dx = \int_0^4 \frac{dg}{\sqrt{g}} dg = [2\sqrt{g}]_0^4 = 2\sqrt{4} - 2\sqrt{0} = 4$$

Example. Evaluate:

$$\int_0^a \frac{du}{\sqrt{b^2-u^2}}$$

Use $u = b \sin x$, then $u'(x) = b \cos x$, using $f(u) = 1/\sqrt{b^2-u^2}$, this becomes $1/b \cos x$. The limits become $u = 0 \Rightarrow x = 0$ and $u = b \Rightarrow x = \sin^{-1}(a/b)$ inserting this into the LHS of (54) shows that:

$$\int_0^a \frac{du}{\sqrt{b^2-u^2}} = \int_0^{\sin^{-1}(a/b)} \frac{b \cos x}{b \cos x} dx = \int_0^{\sin^{-1}(a/b)} dx = \sin^{-1}(a/b)$$

Example. Evaluate:

$$\int_0^a \frac{dx}{1+\sqrt{x}}$$

Let $x = u^2$ then $x'(u) = 2u$ and the limits become $x = 0 \Rightarrow u = 0$ and $x = a \Rightarrow u = \sqrt{a}$, then inserting the values into the LHS of (54) shows that:

$$\begin{aligned}
\int_0^a \frac{dx}{1+\sqrt{x}} &= 2\int_0^{\sqrt{a}} \frac{u}{1+u} du \\
&= 2\int_0^{\sqrt{a}} \frac{1+u-1}{1+u} du \\
&= 2\int_0^{\sqrt{a}} du - \int_0^{\sqrt{a}} \frac{du}{1+u} \\
&= [u - \ln|1+u|]_0^{\sqrt{a}} \\
&= \sqrt{a} - \ln(1+\sqrt{a})
\end{aligned}$$

Example. Evaluate:

$$\int_0^1 x^2\sqrt{x^3+1}dx$$

Let $g(x) = x^3 + 1$ then $g'(x) = 3x^2$, with the limits $g(0) = 1$ and $g(1) = 2$, The integral becomes transformed as:

$$\int_0^1 x^2\sqrt{x^3+1}dx = \frac{1}{3}\int_1^2 \sqrt{u}du = \frac{2}{9}(2\sqrt{2}-1)$$

6 Miscellaneous Methods of Integration

This section is all about the one off techniques which can be useful in certain situations

6.1 Use of Partial Fractions

Integrals of the form:

$$\int_a^b \frac{dx}{(x+\alpha)(x+\beta)}$$

The idea is to use partial fractions to decompose into a simpler integrand. So using partial fractions the integral becomes:

$$\frac{1}{(x+\alpha)(x+\beta)} = \frac{1}{\beta-\alpha}\left(\frac{1}{x+\alpha} - \frac{1}{x+\beta}\right)$$

Then the integral becomes:

$$\int_a^b \frac{dx}{(x+\alpha)(x+\beta)} = \frac{1}{\beta-\alpha}\int_a^b \frac{1}{x+\alpha} - \frac{1}{x+\beta}dx = \frac{1}{\beta-\alpha}\left[\ln\left(\frac{x+\alpha}{x+\beta}\right)\right]_a^b$$

So all the methods of partial fractions can be brought to bare on problems of integration of ratios of polynomials.
Example. Evaluate the indefinite integral:

$$\int \sec x dx$$

The way to proceed with this integral is:

$$\int \sec x dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1 - \sin^2 x}$$

Now use partial fractions on $1 - \sin^2 x = (1 - \sin x)(1 + \sin x)$ to obtain:

$$\frac{1}{(1 - \sin x)(1 + \sin x)} = \frac{1}{2}\left(\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}\right)$$

Inserting this back into the integral:

$$\begin{aligned}
\int \frac{\cos x}{1 - \sin^2 x} &= \frac{1}{2}\int \frac{\cos x}{1 + \sin x} + \frac{\cos x}{1 - \sin x} dx \\
&= \frac{1}{2}\int \frac{\cos x}{1 + \sin x} dx + \frac{1}{2}\int \frac{\cos x}{1 - \sin x} dx \\
&= \frac{1}{2}\ln|1 + \sin x| - \frac{1}{2}\ln|1 - \sin x| + C \\
&= \frac{1}{2}\ln\left|\frac{1 + \sin x}{1 - \sin x}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{(1 + \sin x)(1 + \sin x)}{(1 - \sin x)(1 + \sin x)}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{1 + \sin x}{\cos x}\right|^2 + C \\
&= \ln\left|\frac{1 + \sin x}{\cos x}\right| + C \\
&= \ln|\sec x + \tan x| + C
\end{aligned}$$

6.2 Use of Substitution $t = \tan(x/2)$

Integrals of the form:

$$\int_a^b f(\sin x, \cos x) dx$$

can be easily integrated by the use of a specific substitution. The trick here is to use the substitution:

$$t = \tan\left(\frac{x}{2}\right)$$

or in the notation that we are used to:

$$x = 2\tan^{-1} t$$

Using the double angle formula for $\tan(x/2)$ we see that:

$$\tan x = \frac{2\tan(x/2)}{1-\tan^2(x/2)} = \frac{2t}{1-t^2}.$$

Upon using the identities $1+\tan^2 z = \sec^2 x$ and $1+\cot^2 = \text{cosec}^2 x$ we can write:

$$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$$

Differentiating the original substitution shows that:

$$x'(t) = \frac{2}{1+t^2}$$

Example. Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$$

As suggested use the substitution $x = 2\tan^{-1} t$ then:

$$\frac{1}{2+\sin x} = \frac{1+t^2}{2(t^2+t+1)}$$

The limits become $x = 0 \Rightarrow t = 0$ and $x = \pi/2 \Rightarrow t = 1$, inserting everything into the LHS of (54) shows that:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} &= \int_0^1 \frac{1+t^2}{2(t^2+t+1)} \frac{2}{1+t^2} dt \\
&= \int_0^1 \frac{dt}{t^2+t+1} \\
&= \int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1 \\
&= \frac{2}{\sqrt{3}}(\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\
&= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) \\
&= \frac{\pi}{3\sqrt{3}}
\end{aligned}$$

7 Trapezium Rule

There is a refinement on how the area under the graph or area above the graph, that is computing the integral:

$$\int_a^b f(x)dx.$$

We proceed as usual by choosing a partition $a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b$ but instead of having a rectangular block as an approximation to the area a trapezium is used, as is seen in figure 9. The area of the trapezium between points x_i and x_{i+1} is given by $A_i = (x_{i+1} - x_i)(f(x_i) + f(x_{i+1}))/2$, clearly:

$$A_L(n) \leqslant \sum_{i=1}^{n} A_i \leqslant A_U(n)$$

So it can be seen that if $f(x)$ is integrable and therefore the upper and lower

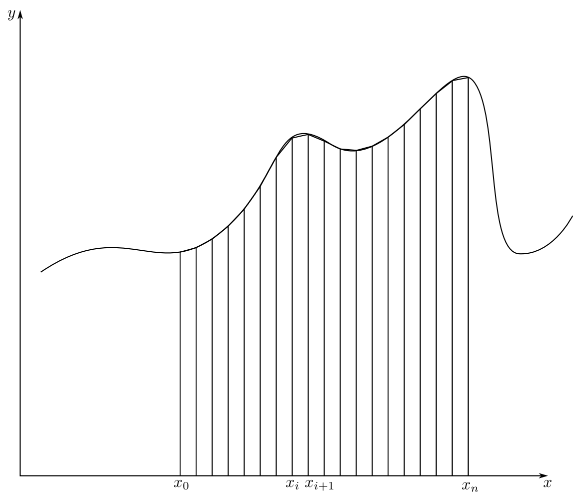

Figure 9: Trapezium Rule

integrals tend to the same number as $n \to \infty$ then $\sum_{i=1}^{n} A_i$ must also tend to the same number. The trapezium rule is meant to be an approximation to the area under (or over) the graph and we can choose the accuracy as we like. For simplicity all the widths of the trapeziums are taken to be the same length, so $h = x_{i+1} - x_i$ for all i, taking n terms shows that:

$$\begin{aligned}
\int_a^b f(x)dx &\approx \sum_{i=0}^{n} A_i \\
&\approx \frac{1}{2}\sum_{i=0}^{n}(f(x_i) + f(x_{i+1}))(x_{i+1} - x_i) \\
&\approx \frac{h}{2}\sum_{i=0}^{n}(f(x_i) + f(x_{i+1})) \\
&\approx \frac{h}{2}(f(x_0) + f(x_1) + f(x_1) + f(x_2) + \cdots + f(x_{n-2}) + \\
&\quad + f(x_{n-1}) + f(x_{n-1}) + f(x_n) \\
&\approx \frac{h}{2}\left(f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right)
\end{aligned}$$

This is the basis of the approximation for the integral. As an example take:

$$\int_0^1 x^2 dx.$$

The value of this integral is 1/3. Let us take 10 strips in our numerical scheme, so the partition will be $0 < 0.1 < 0.2 < \cdots < 0.9 < 1.0$ with $h = 0.1$, then the value of the integral can be approximated as:

$$\begin{aligned}
\int_0^1 x^2 dx &\approx \frac{0.1}{2}(0 + 1 + 2 \times (0.1^2 + 0.2^2 + 0.3^2 + 0.4^2 + 0.5^2 + \\
&\quad + 0.6^2 + 0.7^2 + 0.8^2 + 0.9^2)) = 0.335
\end{aligned}$$

This answer is accurate to within 0.5% which isn't a bad approximation. The larger the value of n the better the approximation, taking $n = 100$ for example yields $h = 0.01$ and the answer to be 0.33335 which is accurate to within 0.005%

8 Simple Integral Equations

An *integral equation* is an equation of the form:

$$g(x)u(x) = f(x) + \int_{a(x)}^{b(x)} K(x,t)u(t)dt \tag{55}$$

The function $K(x,t)$ is called the *kernel* of the integral equation. In general integral equations are rather difficult to solve but there are certain cases where solutions can be found. We take the case where $a(x)$ and $b(x)$ are constant, $g(x) = 1$ and $K(x,t) = \alpha(x)\beta(t)$. With this (55) becomes:

$$u(x) = f(x) + \alpha(x)\int_a^b \beta(t)u(t)dt \tag{56}$$

Now it can be seen that the integral becomes a constant,

$$\int_a^b \beta(t)u(t)dt = \gamma = \text{constant} \tag{57}$$

Inserting (57) into (56) shows that:

$$u(x) = f(x) + \gamma\alpha(x) \tag{58}$$

So this is the solution, all that needs to be done how is to compute the constant γ, This is done by inserting (58) into (56) and obtaining an equation for γ which can be solved. Doing this shows that:

$$u(x) = f(x) + \alpha(x)\int_a^b \beta(t)(f(t) + \gamma\alpha(t))dt \tag{59}$$

A simple example will illustrate the principle:
Example. Solve the equation:

$$u(x) = \cos x + 2x + \int_0^\pi xtu(t)dt$$

We recognise that $f(x) = \cos x + 2x$, $\alpha(x) = x$ and $\beta(t) = t$ and that γ is given by:

$$\gamma = \int_0^\pi tu(t)dt$$

Then $u(x)$ is given by:

$$u(x) = \cos x + (2 + \gamma)x$$

So calculating γ shows that:

$$\begin{aligned}
\gamma &= \int_0^\pi t(\cos t + (2+\gamma)t)dt \\
&= \int_0^\pi t\cos t dt + (2+\gamma)\int_0^\pi t^2 dt \\
&= [t\sin t]_0^\pi - \int_0^\pi \sin t dt + (2+\gamma)\left[\frac{1}{3}t^3\right]_0^\pi \\
&= 0 - [-\cos t]_0^\pi + \frac{\pi^3}{3}(2+\gamma) \\
&= -2 + \frac{\pi^3}{3}(2+\gamma)
\end{aligned}$$

Re-arranging shows that $\gamma = -2$ and the solution is $u(x) = \cos x$.