Exercise Sheet 6 - Solutions

September 17, 2011

1. (a) $y(x) = x^2 + 2x + 1$, so $y(a+h) = (a+h)^2 + 2(a+h) + 1 = a^2 + 2a + 1 + 2ah + h^2 + 2h$, so computing the ratio and taking the limit

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{y(a+h) - y(a)}{h} = \lim_{h\to 0} \frac{2ah + 2h + h^2}{h} = \lim_{h\to 0} \frac{2a + 2 + h}{1} = 2a{+}2$$

   (b) $y(x) = 3x^2 + 8x - 16$, so $y(a+h) = 3(a+h)^2 + 8(a+h) - 16 = 3a^2 + 8a - 16 + 6ah + 3h^2 + 8h$, then computing the ratio and taking the limit

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{y(a+h) - y(a)}{h} = \lim_{h\to 0} \frac{6ah + 3h^2 + 8h}{h} = \lim_{h\to 0} \frac{6a + 8 + 3h}{1} = 6a{+}8$$

2. (a) $y'(x) = (-3)x^{-3-1} + 2 = -3x^{-4} + 2$

   (b) Use the product rule $u = x^{-1}$ and $v = 1 + x^2$, then $y'(x) = -x^{-2}(1 + x^2) + x^{-1}(2x) = 1 - x^{-2}$

   (c) Use the quotient rule $u = \sqrt{x} + 7$ and $v = x^2$ then $y'(x) = (x^2(x^{-\frac{1}{2}}/2) - (\sqrt{x} + 7)(2x))/x^4$

3. (a) let $r(x) = x^2$ then $h^2(x) = r(h(x)) = h \circ r(x)$. The derivative of this is just $h \circ r'(x)h'(x) = 2hh'(x)$

   (b) $(f^2(x) + g^2(x))' = (f^2(x))' + (g^2(x))' = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x)g(x) + 2g(x)(-f(x)) = 0$

4. (a) Plugging in $x = 0$ shows that you obtain $0/0$, so use L'Hopital's rule to obtain:

$$\lim_{x\to 0} \frac{1 - \sqrt{1 + x^2}}{x} = \lim_{x\to 0} \frac{x/\sqrt{1 + x^2}}{1} = 0/1 = 0$$

(b) Plugging $x = 1$, obtains $0/0$, so we use L'Hopital's rule:

$$\lim_{x\to 1} \frac{x^2+x-1}{6x^3-7x^2+1} = \lim_{x\to 0} \frac{2x+1}{18x^2-14x} = \frac{2\times 1}{18-14} = \frac{3}{2}$$

5. (a) Write:

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = a + bx + cx^2 + \cdots$$

Differentiating shows that:

$$-\frac{1}{2}(1+x)^{-\frac{3}{2}} = b + 2cx + \cdots$$

Differentiating once again shows that:

$$\frac{1}{2}\frac{3}{2}(1+x)^{-\frac{5}{2}} = 2x + \cdots$$

Setting $x = 0$ in the first equation shows that $a = 1$, setting $x = 0$ in the second equation shows that $b = -1/2$ and setting $x = 0$ in the third equation shows that $c = 3/8$, and so:

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 + \cdots$$

(b) write:

$$\frac{1}{1+\sqrt{1+x}} = (1+\sqrt{1+x})^{-1} = a + bx + cx^2 + \cdots$$

Differentiating once shows that

$$-(1+\sqrt{1+x})^{-2}\left((1+x)^{-\frac{1}{2}}/2\right) = b + 2cx + \cdots$$

Differentiating again using the product rule:

$$2\left(1+\sqrt{1+x}\right)^{-3}\left(\frac{1}{2}(1+x)^{-\frac{1}{2}}\right) - \left(1+\sqrt{1+x}\right)^{-2}\left(-\frac{1}{4}(1+x)^{-\frac{3}{2}}\right) = 2c + \cdots$$

Setting $x = 0$ in the first equation shows that $a = 1/2$, setting $x = 0$ in the second equation shows that $b = -1/8$ and setting $x = 0$ in the final equation shows that $c = 3/32$, hence:

$$\frac{1}{1+\sqrt{1+x}} = \frac{1}{2} - \frac{x}{8} + \frac{3}{32}x^2$$

6. (a) $y = 4x^3+3x^2-6x-1$, the $y'(x) = 12x^2+6x-6$, to find the turning points set $y'(x) = 0$ to find that $2x^2+x-1 = 0 = (2x-1)(x+1)$, so there are turning points at $x = 1/2$ and at $x = -1$, differentiating again shows that $y''(x) = 6(4x + 6)$, inserting $x = 1/2$ shows that $y''(1/2) > 0$ and so that means that $x = 1/2$ is a minimum. Likewise at $x = -1$, $y''(-1) < 0$, so there is a maximum at $x = -1$

   (b) $y = 9x^{-1} + x$, then differentiating shows that $y'(x) = -9x^{-2} + 1$ and so the turning points are at $y'(x) = 0$ which are given by $x^2 = 9$ and so $x = \pm 3$. Differentiating again shows that $y''(x) = 18x^{-3}$ and so $y''(3) > 0$ and hence $x = 3$ is a minimum, likewise $y''(-3) < 0$ and so $x = -3$ is a maximum.

7. (a) The length of the fence is $80m$ and we have to make a rectangular enclosure, call one side x metres long and the other side y metres long. As the wall will be one side of the enclosure, let it be y metres, then the length of the fence will be $2x + y = 80$, the area of the enclose is given by $A = xy = x(80 - 2x) = 80x - 2x^2$.

   (b) differentiating shows that $a'(x) = 80 - 2x$ and to find the turning point, set $A'(x) = 0$ to obtain $x = 20$ as the turning point. Differentiating again shows that $A''(x_{=} - 4 < 0$ and so the turning point is a maximum, the maximum area will be in that case $a = 80 \times 20 - 2 \times (20)^2 = 800m^2$