Exercise Sheet 7 - Solutions

September 26, 2011

1. (a) Taking the hint, write $\tan x = \sin x / \cos x$ and use the quotient rule with $u = \sin x$ and $v = \cos x$ to obtain:

$$\tan'(x) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

   (b) From a previous question sheet we showed that the derivative of $h^2(x)$ was $2h'(x)h(x)$, applying this to $h(x) = \sin x$, shows that the derivative of $\sin^2 x$ is $2\sin x(\sin' x) = 2\sin x \cos x = \sin 2x$

2. (a) Plugging in 0 to the limit shows that you obtain the limit $0/0$, so we use L'Hopitals rule and differentiate the top and bottom separately to obtain:

$$\lim_{x\to 0} \frac{\tan x - x}{\sin x - x} = \lim_{x\to 0} \frac{\sec^2 x - 1}{\cos x - 1}$$

   Plugging $x = 0$ shows that you get $0/0$ and so we must differentiate again:

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{2\sec x}{-\sin x}$$

   A side calculation shows that $\sec' x = \sin x \sec^2 x$ and so:

$$\lim_{x\to 0} \frac{2\sec x}{-\sin x} = \lim_{x\to 0} \frac{2\sec x \cancel{\sin x} \sec^2 x}{-\cancel{\sin x}} = \lim_{x\to 0} -2\sec^3 x = -2$$

   (b) Plugging in $x = 0$ shows that the limit is $0/0$ and so we must employ L'hopitals rule multiple times.

$$\begin{aligned} \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} &= \lim_{x\to 0} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{3x^2} \\ &= \lim_{x\to 0} -\frac{1}{3}\frac{\cancel{x}\sin x}{x^2} \end{aligned}$$

$$\begin{aligned} &= \lim_{x\to 0} -\frac{1}{3}\frac{\sin x}{x} \\ &= \lim_{x\to 0} -\frac{1}{3}\cos x \\ &= -\frac{1}{3} \end{aligned}$$

(a) We are asked to find all solution in the interval of $\sin^2 2x = 1/2$, taking square roots shows that $\sin 2x = \pm 1/\sqrt{2}$, so we have two equations to solve $\sin 2x = 1/\sqrt{2}$ and $\sin 2x = -1/\sqrt{2}$, let is deal with the first equation $\sin 2x = \pm 1/\sqrt{2}$, this equation has two solutions, write $2x = X$ and solve $\sin X = 1/\sqrt{2}$, the first solution we see is that $x = \pi/4$ is a solution and using the equation $\sin(\pi - X) = \sin X$ and so $\sin(3\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$. As $X = 2x$, this shows that the two solutions are $x = \pi/8, 2\pi/8$. To deal with the second equation $\sin 2x = -1/\sqrt{2}$, not that $\sin(\pi + X) = -\sin X$ so all we do is use $X = \pi/4, 3\pi/4$ in this equation to obtain $X = 5\pi/4, 7\pi/4$ and this leads to $x = 5\pi/8, 7\pi/8$, these are the other two solutions, so all four of them are $x = \pi/8, 3\pi/8, 5\pi/8, 7\pi/8$

(b) Use $\cos^2 2x = 1 - \sin^2 2x$ to show that the equation reduces to $\sin^2 2x = 3/4$ which gives two equations $\sin 2x = \sqrt{3}/2$ and $\sin 2x = -\sqrt{3}/2$, so we are lead to the previous calculation, so let $X = 2x$ and solve the equation $\sin X = \sqrt{3}/2$ to find that $X = \pi/3$ and also $\sin(\pi - \pi/3) = \sin(\pi/3) = \sqrt{3}/2$ and so $X = \pi/3, 2\pi/3$ and hence $x = \pi/6, \pi/3$. Using $\sin(\pi + X) = -\sin X$ shows that the other solutions are $X = \pi/3 + \pi, 2\pi/3 + \pi$ and hence $x = 2\pi/3, 5\pi/6$

3.

$$\begin{aligned} \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \\ &= \frac{1}{\cos\theta\sin\theta} \\ &= \frac{2}{2\sin\theta\cos\theta} \\ &= \frac{2}{\sin 2\theta} \end{aligned}$$

4.

$$(\sec x - \cos x)(\mathrm{cosec} x - \sin x) = \left(\frac{1}{\cos x} - \cos x\right)\left(\frac{1}{\sin x} - \sin x\right)$$

$$
\begin{aligned}
&= \frac{(1-\sin^2 x)(1-\cos^2 x)}{\cos x \sin x} \\
&= \frac{1-(\sin^2 x+\cos^2 x)+\sin^2 x\cos^2 x}{\cos x \sin x} \\
&= \frac{1-1+\cos^2 x\sin^2 x}{\cos x \sin x} \\
&= \cos x \sin x
\end{aligned}
$$

Likewise:

$$
\begin{aligned}
\frac{1}{\tan x+\cot x} &= \frac{1}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}} \\
&= \frac{1}{\frac{\sin^2 x+\cos^2 x}{\sin x \cos x}} \\
&= \frac{1}{\frac{1}{\sin x \cos x}} \\
&= \cos x \sin x
\end{aligned}
$$

So both sides are the same expression and therefor the identity is the same.