Exercise Sheet 8 - Solutions

September 30, 2011

1. Note that $5^{x+1} = 5 \cdot 5^x$ and so on multiplying by 5^x the equation becomes $10 \cdot 5^{2x} - 5^x - 3 = 0$. Let $u = 5^x$ and the equation becomes $10u^2 - u - 3 = 0$, use the quadratic formula to solve this to obtain:

$$u = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(10)(-3)}}{20} = \frac{1 \pm \sqrt{121}}{20} = \frac{1 \pm 11}{20}$$

   So there are two solutions $u = 3/5$ and $u = -1/2$ as $u = 5^x$, all negative solutions are ruled out, so the only solution is $u = 3/5 = 5^x$, taking logs of this shows that $x = \log_5(3/5) = \log_5(3 \cdot 5^{-1}) = \log_5(3) + \log_5(5^{-1}) = -1 + \log_5 3$ so $k = 3$

2. Taking the hint, write $g(x) = n \ln x$ and $g'(x) = n/x$ and so as $f(x) = x^n = e^{n \ln x}$ and $f'(x) = e^{g(x)} g'(x)$ as we were told, $f'(x) = x^n \cdot (n/x) = nx^{n-1}$

3. As $f(x) = g \circ h(x)$ with $h(x) = \ln x$, the derivative according to the chain rule is $f'(x) = g \circ h'(x) g'(x) = g'(x)/g(x)$

4. (a) Plugging in $x = 0$ straight away shows that one obtains the limit of $0/0$ and one must apply L'hopital's rule. We can be clever about how many times we can use the rule as the denominator contains an x^3, so that will imply that we will need to apply the rule 3 times:

$$\begin{aligned} \lim_{x \to 0} \frac{e^x - e^{-x} - 2\sin x}{x^3} &= \lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{3x^2} \\ &= \lim_{x \to 0} \frac{e^x - e^{-x} + 2\sin x}{6x} \\ &= \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x}{6} \end{aligned}$$

Now inserting $x = 0$, shows that the limit is $(1+1+2)/6 = 4/6 = 2/3$, the same result can be obtained by using the MacLaurin series expansion for the functions involved, so $e^x = 1+x+x^2/2+x^3/6+\cdots$, $e^{-x} = 1 - x + x^2/2 - x^3/6 + \cdots$ and $\sin x = x - x^3/6 + \cdots$, inserting these into the quotient shows that:

$$\begin{aligned}\lim_{x\to 0}\frac{e^x - e^{-x} - 2\sin x}{x^3} &= \lim_{x\to 0}\frac{1+x+x^2/2+x^3/6-(1-x+x^2/2-x^3/6)}{x^3} - \\ &\quad - \frac{2(x-x^3/6)}{x^3} \\ &= \frac{2x+x^3/3-2(x-x^3/6)}{x^3} \\ &= \frac{2x^3/3}{x^3}\end{aligned}$$

The x^3 terms cancel and the limit becomes $2/3$ as before

(b) Examining the limit by placing $x = 1$ into the denominator and numerator to show that one obtains $0/0$ as a limit. The way forward is to use L'Hopital's rule.

$$\lim_{x\to 1}\frac{\ln x}{x^2-1} = \lim_{x\to 1}\frac{1/x}{2x}$$

Inserting $x = 1$ shows that the limit is $1/2$. The MacLaurin series for $\ln(1+x)$ is given by: $\ln(1+x) = x - x^2/3 + x^3/3 + \cdots$ replacing $1 + x$ by X say, shows that $\ln X = (X-1) - (X-1)^2 + (X-1)^3/3 + \cdots$ and so the denominator and numerator become:

$$\begin{aligned}\frac{\ln x}{x^2-1} &= \frac{\ln x}{(x-1)(x+1)} \\ &= \frac{(x-1)-(x-1)^2/2+\cdots}{(x-1)(x+1)} \\ &= \frac{\cancel{(x-1)}(1-(x-1)/2+\cdots)}{(\cancel{(x-1)}(x+1)} \\ &= \frac{1-(x-1)/2+\cdots}{(x+1)}\end{aligned}$$

Upon taking the limit as $x \to 1$ the limit can be seen to be $1/2$ as before.

5. (a) Let $u(x)) = 2^x = e^{\ln 2^x} = e^{x\ln 2}$ and $v(x) = \ln x$ then $u'(x) = 2^x \ln 2$ and $v'(x) = 1/x$ and so from the product rule $f'(x) = v(x)u'(x) + u(x)v'(x)$, hence:

$$f'(x) = 2^x(\ln 2)(\ln x) + 2^x/x$$

(b) Let $u(x) = e^x$ and $v(x) = x + \ln x$, then $u'(x) = e^x$ and $v'(x) = 1 + 1/x = x/(1+x)$ and so from the quotient rule $g'(x) = (v(x)u'(x) - u(x)v'(x))/(v(x))^2$, hence:

$$g'(x) = \frac{e^x(x + \ln x) - xe^x/(1+x)}{(x + \ln x)^2}$$

6. Write $2^x = a + bx + cx^2 + \cdots$, then inserting $x = 0$ to find that $2^0 = a = 1$, so $2^x = 1 + bx + cx^2 + \cdots$. Differentiating once shows that $2^x \ln 2 = b + 2cx + \cdots$, inserting $x = 0$ shows that $2^0 \ln 2 = \ln 2 = b$, and so $2^x = 1 + x \ln 2 + cx^2 + \cdots$. Differentiating this twice shows that $2^x (\ln 2)^2 = 2c + \cdots$, setting $x = 0$ shows that $2^0 (\ln 2)^2 = 2c$ and so:

$$2^x = 1 + x \ln 2 + \frac{(\ln 2)^2}{2} x^2 + \cdots$$