3D Waves in Electrohydrodynamics

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Governing Equations - Hydrodynamics

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \qquad \text{velocity potential}$$

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w \quad z = \eta \qquad \text{free surface}$$

$$\frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial y} - \frac{\partial \varphi}{\partial z} = 0 \quad z = -h + f(x) \qquad \text{topography}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\nabla \varphi|^2 + \frac{p}{\rho} + g\eta = f(t) \quad z = \eta \qquad \text{Bernoulli equation}$$

The pressure comes through the boundary between the conducting fluid and air via the Young-Laplace equation:

$$[\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]_a^f = \sigma \nabla \cdot \hat{\mathbf{n}}$$

This is then inserted into unsteady Bernoulli equation which is how the electric field and the fluid interact and used as a boundary condition. The equations govening the electric field are then

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= 0 \qquad \text{electric potential} \\
\frac{\partial \eta}{\partial x}\frac{\partial V}{\partial x} - \frac{\partial V}{\partial z} &= 0 \qquad \text{Tangential component continuous} \\
\frac{\partial V}{\partial z} &\to -E_0 \qquad \text{as} \quad z \to \infty
\end{aligned}$$

Nondimensionalisation

The nondimensional length scales used are the following:

$$x = h\hat{x}, \quad y = h\hat{y}, \quad z = h\hat{z}, \quad \eta = h\hat{\eta}$$

The other nondimensional groups are:

$$t = \sqrt{\frac{\rho h^3}{\sigma}}\hat{t} \quad \varphi = \sqrt{\frac{\sigma h}{\rho}}\hat{\varphi}, \quad V = E_0 h \hat{V}$$

This yields three nondimensional groups, B, E_b and $\varepsilon << 1$ defined as:

$$B = \frac{gh^2\rho}{\sigma}, \quad E_b = \frac{\epsilon V_0^2}{h\sigma}, \quad \varepsilon = \frac{\rho U^2}{h\sigma}$$

Linear Theory I

Expand according to:

$$\begin{aligned} p &= \varepsilon p_1 + o(\varepsilon) \\ \eta &= \varepsilon \eta_1 + o(\varepsilon) \\ \varphi &= Ux + \varepsilon \varphi_1 + o(\varepsilon) \\ V &= -z + \varepsilon V_1 + o(\varepsilon) \end{aligned}$$

The dispersion relation is:

$$U^2 = (B\mu - E_b\mu^2 + \mu^3)\frac{\tanh \mu}{k^2} \quad \mu = \sqrt{k^2 + l^2}$$

Linear Theory II

The linear equations are given by:

$$
\begin{aligned}
\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}+\frac{\partial^2\varphi_1}{\partial z^2} &= 0 \quad \text{on} \quad -1\leqslant z\leqslant 0 \\
\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}+\frac{\partial^2 V_1}{\partial z^2} &= 0 \quad \text{on} \quad z>0 \\
U\frac{\partial\eta_1}{\partial x} &= \frac{\partial\varphi_1}{\partial z} \quad \text{on} \quad z=0 \\
U\frac{\partial\varphi_1}{\partial x}+B\eta_1+p_1+E_b\frac{\partial V_1}{\partial z} &= \frac{\partial^2\eta_1}{\partial x^2}+\frac{\partial^2\eta_1}{\partial y^2}+K \\
\frac{\partial V_1}{\partial x} &= \frac{\partial\eta_1}{\partial x} \quad \text{on} \quad z=0 \\
\frac{\partial\varphi_1}{\partial z} &= 0 \quad \text{on} \quad z=-1 \\
\lim_{z\to\infty}\frac{\partial V_1}{\partial z} &\to 0
\end{aligned}
$$

Linear Theory III

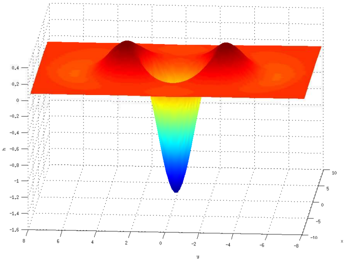
Let the pressure distribution be given by:

$$p = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{p} e^{i(kx+ly)} dkdl$$

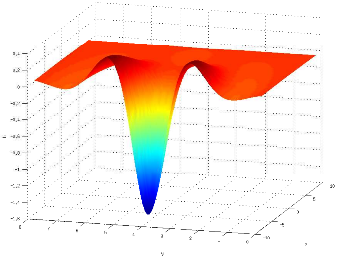
The free surface is given by:

$$\eta(x, y) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\mu e^{i(kx+ly)} \hat{p} \tanh \mu}{k^2 U^2 - (B\mu - E_b \mu^2 + \mu^3) \tanh \mu} dkdl$$

Linear Theory IV



(a) Full Profile



(b) Half Profile

Infinite Depth I

Following Vanden-Broeck, replace the two boundary conditions as:

$$\frac{\partial \varphi}{\partial x} \rightarrow U \quad \text{as} \quad z \rightarrow \infty$$
$$\frac{\partial \varphi}{\partial z} \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty$$

Then expand the variables as:

$$\begin{aligned} \varphi &= Ux + \varepsilon \varphi_1 + o(\varepsilon) \\ \eta &= \varepsilon \eta_1 + o(\varepsilon) \\ V &= -E_0 z + \varepsilon V_1 + o(\varepsilon) \end{aligned}$$

Infinite Depth II

The linear equations become:

$$
\begin{aligned}
\nabla^2 \varphi_1 &= 0 \\
\nabla^2 V_1 &= 0 \\
U \frac{\partial \eta_1}{\partial x} &= \frac{\partial \varphi_1}{\partial z} \\
\frac{\partial V_1}{\partial x} &= E_0 \frac{\partial \eta_1}{\partial x} \\
U \frac{\partial \varphi_1}{\partial x} + g\eta_1 + \frac{P}{\rho} + \frac{\epsilon E_0}{\rho} \frac{\partial V_1}{\partial z} &= \frac{\sigma}{\rho} \left(\frac{\partial^2 \eta_1}{\partial x^2} + \frac{\partial^2 \eta_1}{\partial y^2} \right) \\
\frac{\partial \varphi}{\partial x} &\to U \quad \text{as} \quad z \to -\infty \\
\frac{\partial \varphi}{\partial z} &\to 0 \quad \text{as} \quad z \to -\infty
\end{aligned}
$$

Infinite Depth III

Defining a pressure in the same way as the finite depth case:

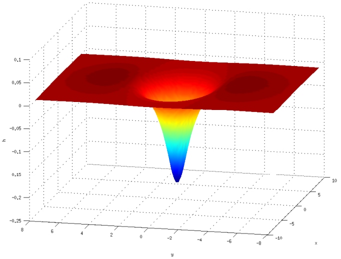
$$p = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{p}e^{i(kx+ly)}dkdl$$

$$\hat{\eta}_1(\hat{x},\hat{y}) = \frac{1}{8\pi^2}\int_{\mathbb{R}^2} \frac{\nu e^{-\frac{\nu^2}{20}}e^{i(\gamma\hat{x}+\delta\hat{y})}}{\gamma^2-\nu+\mu_1\nu^2-\mu_2\nu^3}d\gamma d\delta \quad \nu = \sqrt{\gamma^2+\delta^2}$$

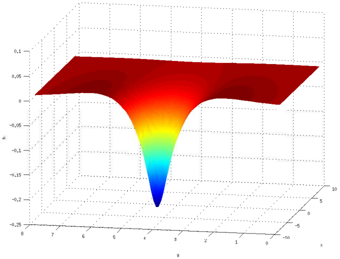
where:

$$\mu_1 = \frac{\epsilon E_0^2}{\rho U^2} \quad \mu_2 = \frac{\sigma g}{\rho U^4}, \quad \hat{x} = xgU^{-2},$$

$$\hat{y} = ygU^{-2}, \quad \eta_1 = g\hat{\eta}U^{-2} \quad \alpha = \frac{\sigma g}{\rho U^2}$$



(c) Full Profile



(d) Half Profile

Blow Up

There will be a zero of the denominator in the expression for the free surface when $l = 0$ and when $1 + \mu_1 = 2\sqrt{\mu_2}$

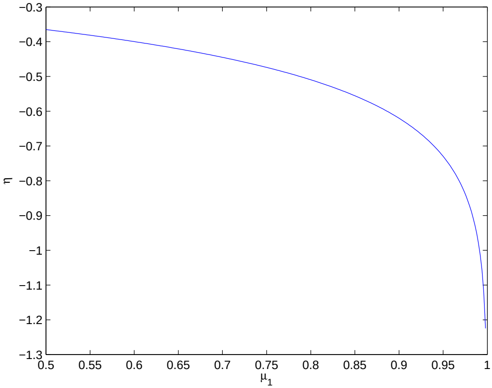


Figure : Demonstration of blow up phenomena

In order to do this, the following scaling is used:

$$x = \lambda \hat{x}, \quad y = \mu \hat{y}, \quad t = \frac{\lambda}{c_0}\hat{t}, \quad \eta = a\hat{\eta} \quad \varphi = \frac{g\lambda a}{c_0}\hat{\varphi} \tag{1}$$

$$V = \lambda E_0 \hat{V} \quad z^{(1)} = h\hat{z} \quad z^{(2)} = \lambda \hat{Z}, \tag{2}$$

where $c_0 = \sqrt{gh}$, a is the typical amplitude of the waves, and λ is the typical wavelength. Define the parameters:

$$\alpha = \frac{a}{h}, \quad \beta = \frac{h^2}{\lambda^2}, \quad \gamma = \frac{\lambda^2}{\mu^2} \tag{3}$$

set $\alpha = \varepsilon$, $\beta = \varepsilon$ and $\gamma = \varepsilon$. Make the transformation:

$$\begin{aligned} T &= \varepsilon \hat{t} \\ X &= \hat{x} - \hat{t} \end{aligned}$$

Weakly Nonlinear II

Expanding the variables as:

$$\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + o(\varepsilon^2) \quad (4)$$

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + o(\varepsilon^2) \quad (5)$$

$$V_1 = Y + \varepsilon^{3/2}V_1 + o(\varepsilon^{3/2}) \quad (6)$$

The equation becomes:

$$\frac{\partial}{\partial X}\left[\frac{\partial\eta_0}{\partial T} + \frac{1}{2}\left(\frac{1}{3} - B\right)\frac{\partial^3\eta_0}{\partial X^3} + \frac{\partial P}{\partial X} + \frac{3}{2}\eta_0\frac{\partial\eta_0}{\partial X} + \frac{\hat{E}_b}{2}\frac{\partial^2 V}{\partial X\partial Z}\right] + \\ + \frac{1}{2}\frac{\partial^2\eta_0}{\partial y^2} = 0 \quad (7)$$

Weakly Nonlinear III

To find the electric potential term, use the Green's function for the 3D Laplace equation in the half space reduces to a 2D Laplace equation

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (8)$$

To find $\partial_z V_1$ one can use the Hilbert transform to find that:

$$\partial_Z V = \mathscr{H}(\partial_X V) \qquad (9)$$

The resulting equation is then:

$$\frac{\partial}{\partial x}\left[\frac{1}{c_0}\frac{\partial \eta}{\partial x} + \frac{3}{2h}\eta\frac{\partial \eta}{\partial x} + \frac{h^2}{2}\left(\frac{1}{3} - B\right)\frac{\partial^3 \eta}{\partial x^3} + \frac{1}{2\rho g}\frac{\partial p}{\partial x} + \frac{E_b h}{2}\mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right)\right] + \frac{1}{2}\frac{\partial^2 \eta}{\partial y^2} = 0 \qquad (10)$$

KP Around $B = 1/3$

This time set $\alpha = \varepsilon$, $\beta = \varepsilon$ and $\gamma = \varepsilon^2$. Make the transformation:

$$\begin{aligned} T &= \varepsilon^2 \hat{t} \\ X &= \hat{x} - \hat{t} \end{aligned}$$

Expanding the variables as:

$$\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + o(\varepsilon^2) \quad (11)$$

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + o(\varepsilon^3) \quad (12)$$

$$V_1 = Y + \varepsilon^{5/2} V_1 + o(\varepsilon^{5/2}) \quad (13)$$

Final equation

$$\frac{\partial}{\partial x}\left[\frac{1}{c_0}\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}+\frac{h^4}{90}\frac{\partial^5 \eta}{\partial x^5}+\frac{3}{2h}\eta\frac{\partial \eta}{\partial x}-\frac{h^2}{2}\left(B-\frac{1}{3}\right)\frac{\partial^3 \eta}{\partial x^3}+\right.$$
$$\left.+\frac{E_b h}{2}\mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right)+\frac{1}{2\rho g}\frac{\partial p}{\partial x}\right]+\frac{1}{2}\frac{\partial^2 \eta}{\partial y^2}=0 \tag{14}$$

Fully Nonlinear Results I

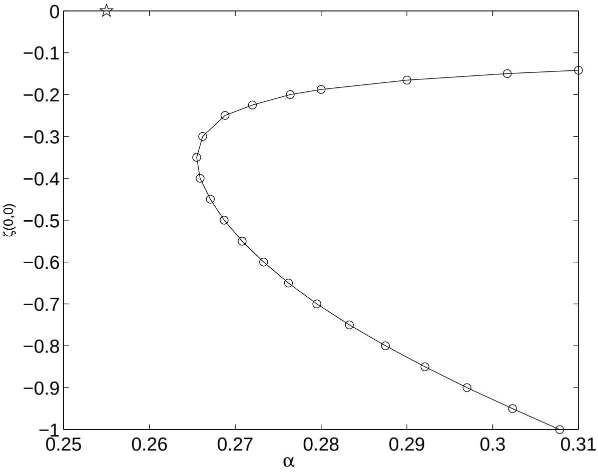


Figure : Blow-up Resolved

Fully Nonlinear Results II

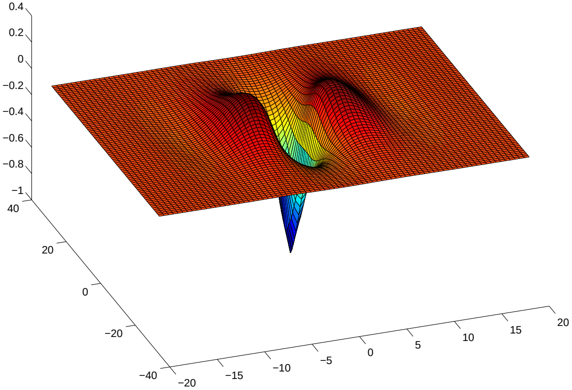


Figure : Typical Profile