VISCO-POTENTIAL FLOWS IN ELECTROHYDRODYNAMICS

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Abstract. In this study we consider the problem of the interface motion under the capillary–gravity and an external electric force. The infinitely deep fluid layer is assumed to be viscous, perfectly conducting and the flow to be incompressible. The weak viscous effects are introduced using the Helmholtz–Leray decomposition and the visco-potential flow approach. The electric charge distributions above and on the free surface are considered. Finally, we derive some linearized analytical solutions for the free surface elevation shape under the localised pressure distribution and the combined action of the forces mentioned herein above.

Key words and phrases: electro-hydrodynamics; free surface flows; viscous dissipation; potential flow

Contents

1. Introduction

Liquid thin films are a common occurrence in the fields of biology and engineering under the guise of coating flows, and have been subject to intensive study. Instabilities and the ensuing dynamics associated to the liquid film can by caused by a number of effects including interfacial instabilities due to surface tension variations (e.g. Marangoni instabilities), gravitational instabilities (such as Rayleigh–Taylor) as well as instabilities

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caused by external fields like electric or magnetic ones. Instabilities of the interface can also be due to topography or a moving pressure distribution (as it is considered in this study), but an external electric field may lead to the stabilization of the interface. More precisely, depending on the asymptotic form of the electric field in question, its effect can be either stabilizing [23] or destabilising [21].

The study of electro-capillary waves was first initiated by G.I. Taylor *et al.*[21] for 3D waves with a prescribed disturbance of the interface. On the other hand, in modern investigations the disturbance is determined according to an applied external electric field with the free surface profile is calculated by solving the electro-hydrodynamic formulation [19]. The work carried out by Hunt [11] focused on forced waves in electrohydrodynamics using the methods set out in [19, 26] to examine linear and weakly nonlinear free surface flows in 2D and weakly 3D (Kadomtsev-Petviashvili-type models). The method proposed in the present study takes into account the viscous effects [24, 25, 22, 23] by keeping the *simplicity* of the potential flow approach. For the sake of the clarity of the exposition, we illustrate this method in 2D infinite depth case.

The theory of visco-potential flows probably originates from the pioneering work of J. Boussinesq (1895) [1] who estimated the water wave amplitude decay due to the effect of viscosity in the linear approximation. Then, this research has been continued in the 70's in the context of nonlinear long wave models [18, 14]. Later, Kit & Shemer (1989) [15] developed a theoretical model which allows the estimation of the wave energy dissipation which included the friction effect at the bottom and lateral walls in a rectangular wave tank. The potential flows of viscous fluids were also thoroughly investigated theoretically by D. Joseph and his collaborators [13, 12]. The visco-potential formulation in the deep water case was derived by Dias *et al.* (2008) [5] and generalized to the finite depth case in [16] and later in [8, 7]. This formulation was validated experimentally and numerically for the practically important case of solitary wave propagation in [17]. More recently, the damping rates for various dissipative operators were investigated numerically in [2, 20]. Finally, the asymptotic long time behaviour of some visco-potential models were found and justified analytically in [4, 3, 9].

The present manuscript is organized as follows. After a brief introduction, in Section 2 we present the derivation of the electrohydrodynamic visco-potential formulation. An analytical solution to the linearized formulation is shown in Section 3 for several values of problem parameters. A particular case of the electrohydrodynamic problem where the charge is distributed on the interface is considered in Appendix A. Finally, the main conclusions and perspectives of this study are outlined in Section 4.

2. Mathematical Model

Consider an infinitely deep channel of a perfectly conducting weakly viscous and incompressible fluid (referred to as the heavy fluid) and is in the region $\{(x, y) | \, x \in \mathbb{R}, \, y \leqslant \eta(x, t)\}$. The flow is assumed to be exactly two-dimensional. A Cartesian coordinate system (x, y) is introduced, with y pointing vertically upwards. The interface between two fluids is given by $y = \eta(x, t)$. The sketch of the physical domain is given on Figure 1. Below we will

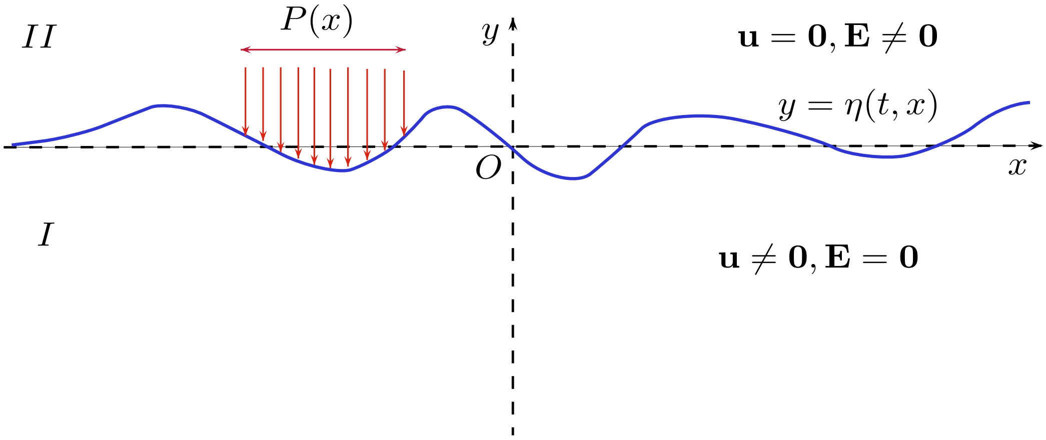

Figure 1. *Sketch of the physical domain considered in this study.*

adopt the so-called free surface assumption and the light fluid will enter into equations only through the electric forces.

The velocity field in the heavy fluid in region I is then given as $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + 0\mathbf{k}$. We assume that there is an electric field in region II $\{(x, y)|\, x \in \mathbb{R},\ y > \eta(x, t)\}$. The electric field $\mathbf{E}$ satisfies the equations $\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$ and, thus, can be written in the following potential form:

$$\mathbf{E} \equiv -\boldsymbol{\nabla} V.$$

Since there are no free charges in the light fluid, the Gauss's law states that $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, and hence the governing equation for the electric potential reads:

$$\boldsymbol{\nabla}^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad y > \eta(x, t).$$

This equation is completed by appropriate boundary conditions. At infinity we require the following asymptotic behaviour of the solution $V(x, y)$:

$$V(x, y) \to -E_0 y \qquad \text{as} \qquad y \to +\infty$$

The usual linear incompressible Navier-Stokes equations are used and use the Helmholtz decomposition [5, 8] on the velocity vector $\mathbf{u} = (u, v, 0)$

$$\mathbf{u} = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla} \times \mathcal{A}, \tag{2.1}$$

with $\mathcal{A} = (0, 0, \psi)$. The idea of the approach taken here is to say that the motion is mainly potential flow but with a small non-potential part, represented by the function ψ i.e. $\|\psi\|_{L^2} \ll \|\varphi\|_{L^2}$.[1] Along with a small viscosity, ν. The decomposition of the velocity is inserted into the linear Navier-Stokes equations to end up with two separate equations for φ and ψ

$$\frac{\partial \psi}{\partial t} = \nu \boldsymbol{\nabla}^2 \psi, \quad \nabla^2 \varphi = 0$$

[1] The L^2 norn is given by:

$$\|f\|_{L^2} = \left(\int_{\mathbb{R}^2} |f(x, y)|^2 dx dy \right)^{\frac{1}{2}}$$

Moreover, in 2D the velocity field $\mathbf{u}$ can be simply expressed as:

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}.$$

The use of decomposition for the velocity and the governing equation for ψ yields a Bernoulli equation in the usual way:

$$\frac{\partial\varphi}{\partial t} + \frac{p_I}{\rho} + gy = C. \tag{2.2}$$

2.1. Boundary Conditions

In contrast to some previous studies which considered the linear case, in this section the derivations of the fully nonlinear equations will be presented.

The capillary and electric effects come into the problem through the Young– Laplace equation at the interface between two media:

$$\bigl[\hat{\mathbf{n}}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}\bigr]_1^2 = \sigma\boldsymbol{\nabla}\cdot\hat{\mathbf{n}}, \tag{2.3}$$

where the brackets $[\cdot]$ denotes evaluation at each side of the interface $y = \eta(x,t)$ and the unit normal $\hat{\mathbf{n}}$ (pointing from the lower fluid to the upper fluid) and unit tangent vector are given by

$$\hat{\mathbf{n}} = \frac{(-\eta_x, 1)}{\sqrt{1+\eta_x^2}}, \quad \hat{\mathbf{t}} = \frac{(1, \eta_x)}{\sqrt{1+\eta_x^2}} \tag{2.4}$$

The stress tensor take on two different forms depending upon which region is under consideration. In region I,

$$T_{ij} = -p\delta_{ij} + \tau_{ij},$$

where p is the pressure as defined above and δ_{ij} is the Kronecker delta symbol. and the tensor τ_{ij} corresponds to the viscosity. The stress tensor in region II is given by

$$T_{ij} = -P\delta_{ij} + \Sigma_{ij},$$

where P is the pressure distribution *on* the interface and Σ_{ij} corresponds to the electric field in region II. Two constitutive components of the viscous and electric stresses are correspondingly:

- $\tau_{ij} = \mu\bigl(\partial_j u_i + \partial_i u_j\bigr)$,
- $\Sigma_{ij} = \varepsilon_p\Bigl(E_iE_j - \frac{1}{2}\delta_{ij}E_kE_k\Bigr)$,

where $\mu = \rho\nu$ is the dynamic viscosity and ε_p is the electric permittivity. Recall that the Young–Laplace condition (2.3) involves the projection of the stresses onto the normal direction $\hat{\mathbf{n}}$ to the interface. In order to avoid cumbersome expressions, we will perform the computations of $\hat{\mathbf{n}}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}$ by parts. The viscous part of the stress tensor $\mathbf{T}$ gives:

$$\hat{\mathbf{n}}\cdot\boldsymbol{\tau}\cdot\hat{\mathbf{n}} = \hat{n}_i\tau_{ij}\hat{n}_j = \hat{n}_1^2\tau_{11} + \hat{n}_2^2\tau_{22} + 2\hat{n}_1\hat{n}_2\tau_{12} = \frac{\nu}{1+\eta_x^2}\Bigl[\eta_x^2(\varphi_{xx}+\psi_{xy}) - 2\eta_x(2\varphi_{xy}+\psi_{yy}-\psi_{xx}) + 2(\varphi_{yy}-\psi_{xy})\Bigr].$$

The same expansion can be also done for the Faraday stress tensor Σ:

$$\hat{\mathbf{n}}\cdot\Sigma\cdot\hat{\mathbf{n}} = \hat{n}_1^2\Sigma_{11} + \hat{n}_2^2\Sigma_{22} - 2\hat{n}_1\hat{n}_2\Sigma_{12},$$

where

$$\Sigma_{11} = \frac{\varepsilon_p}{2}(V_x^2 - V_y^2), \quad \Sigma_{12} = \varepsilon_p V_x V_y, \quad \Sigma_{22} = -\frac{\varepsilon_p}{2}(V_x^2 - V_y^2).$$

Where the approximation $\|\psi\|_{L^2} \ll \|\varphi\|_{L^2}$ was used along with a small viscosity.

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + g\eta + \frac{P}{\rho} - \frac{1}{\rho(1+\eta_x^2)}\Big[\eta_x^2\Sigma_{11} + 2\eta_x\Sigma_{12} + \Sigma_{22}\Big] + \frac{\nu}{1+\eta_x^2}\Big(\eta_x^2\varphi_{xx} - 2\eta_x\varphi_{xy} + 2\varphi_{yy}\Big) = \frac{\sigma}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}, \quad \text{on} \quad y = \eta(x,t).$$

The derivation of the corrected kinematic boundary condition in the framework of the visco–potential free surface flows uses the linearised form of the tangential stress conditions:

$$T_{12} = 0, \quad \text{at} \quad y = \eta(x,t),$$

Since the electric component of the stress tensor $\mathbf{T}$ is nonlinear, after the linearisation this effect disappears. This results in the equation:

$$2\partial_x\partial_y\varphi + \partial_y^2\psi - \partial_x^2\psi = 0 \tag{2.5}$$

The case of non-zero stress shall be considered in future work. Equation (2.5) is used in derivation of the modification of the free surface equation in the presence of a small viscosity [16, 5, 8]:

$$\frac{\partial\eta}{\partial t} = \frac{\partial\varphi}{\partial y} + 2\nu\frac{\partial^2\eta}{\partial x^2}.$$

2.2. Fully Nonlinear Formulation

Now we can write down the set of fully nonlinear equations which govern the motion of the viscous fluid under the action of an exterior electric force:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0, \qquad -\infty < y \leqslant \eta(x,t)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad \eta(x,t) < y < +\infty$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\varphi}{\partial y} + 2\nu\frac{\partial^2\eta}{\partial x^2}, \qquad y = \eta(x,t)$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + g\eta + \frac{P}{\rho} - \frac{1}{\rho(1+\eta_x^2)}\Big[\eta_x^2\Sigma_{11} + 2\eta_x\Sigma_{12} + \Sigma_{22}\Big] + \frac{\nu}{1+\eta_x^2}\Big(\eta_x^2\varphi_{xx} - 2\eta_x\varphi_{xy} + 2\varphi_{yy}\Big) = \frac{\sigma}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}, \quad \text{on} \quad y = \eta(x,t).$$

$$\frac{\partial V}{\partial x} + \frac{\partial\eta}{\partial x}\frac{\partial V}{\partial y} = 0, \quad y = \eta(x,t) \tag{2.6}$$

$$V \to -E_0 y \qquad y \to +\infty,$$

$$\frac{\partial\varphi}{\partial y} \to 0, \qquad y \to -\infty$$

This set of equations will be used below to derive some linearised solutions to the electro-hydrodynamic problem of weakly viscous fluids. A particular case of this problem when the charge is distributed on the interface is discussed in Appendix A.

Remark 1. *The formulation presented above was done in deep water approximation. The generalization to the finite depth case can be done relatively easily. The usual bottom impermeability condition is modified to include a nonlocal term in time which is due to the presence of a boundary layer at the bottom. the details can be found in* [16, 8, 7]*. For the same reasons, the dispersion relation of the visco-electro-hydrodynamic problem coincides with the classical visco-potential formulation* [7, 6]*.*

2.3. Linear Wave Theory on a Uniform Flow

The method of derivation in this case for infinite depth is given in [26] section 4.2.2. It is assumed that there is a uniform flow with speed U, and choose the asymptotic expansion as follows:

$$\begin{aligned}
\varphi &= Ux + \varepsilon\varphi_1 + o(\varepsilon) \\
\eta &= \varepsilon\eta_1 + o(\varepsilon) \\
V &= -E_0 y + \varepsilon V_1 + o(\varepsilon) \\
p &= \varepsilon p_1 + o(\varepsilon)
\end{aligned}$$

The linearisation of the system of equations presented in Section 2.2 reads:

$$\begin{aligned}
\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial y^2} &= 0 \quad -\infty < y \leqslant 0 \\
\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} &= 0 \quad 0 \leqslant y < +\infty \\
U\frac{\partial\eta_1}{\partial x} &= \frac{\partial\varphi_1}{\partial y} + 2\nu\frac{\partial^2\eta_1}{\partial x^2} \quad y = 0 \\
U\frac{\partial\varphi_1}{\partial x} + \frac{p_1}{\rho} + g\eta_1 + \frac{\varepsilon_p E_0^2}{\rho}\frac{\partial V_1}{\partial y} + 2\nu\frac{\partial^2\varphi_1}{\partial y^2} &= \frac{\sigma}{\rho}\frac{\partial^2\eta_1}{\partial x^2} \quad y = 0 \\
\frac{\partial V_1}{\partial x} &= E_0\frac{\partial\eta_1}{\partial x} \\
\frac{\partial\varphi_1}{\partial y} &\to 0 \quad y \to -\infty \\
\frac{\partial V_1}{\partial y} &\to 0 \quad y \to +\infty
\end{aligned} \tag{2.7}$$

The obtained linear system will be understood using the Fourier analysis:

$$\varphi_1(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}_1 \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \quad V_1(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{V}_1 \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k,$$
$$\eta_1(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\eta}_1 \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \quad p_1(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{p}_1 \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

The solutions for $\varphi_1(x, y)$ and $V_1(x, y)$ can be easily obtained:

$$\varphi_1(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} A(k)\mathrm{e}^{|k|y}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \quad V_1(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} B(k)\mathrm{e}^{-|k|y}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

By satisfying the boundary conditions one can find the unknown functions $A(k)$ and $B(k)$:

$$A(k) = (\mathrm{i}U + 2\nu k)\,\mathrm{sgn}(k)\hat{\eta}_1(k), \qquad B(k) = E_0\hat{\eta}_1(k).$$

Inserting all the elements into equation (2.7) allows us to find the Fourier transform of the free surface elevation:

$$\hat{\eta}_1(k) = \frac{\hat{p}_1(k)}{\rho U^2}\Big[k\,\mathrm{sgn}(k) - \frac{4\mathrm{i}k^2\nu\,\mathrm{sgn}(k)}{U} - \frac{g}{U^2} + \frac{\varepsilon_p E_0^2}{\rho U^2}|k| - \frac{4\nu^2k^3\,\mathrm{sgn}(k)}{U^2} - \frac{\sigma}{\rho U^2}k^2\Big]^{-1}.$$

For brevity denote $s_k = \mathrm{sgn}(k)$ and note that $ks_k = |k|$. Following section 4.2.2 in [26], a Gaussian distribution distribution is used:

$$p_1(x) = \frac{\rho U^2}{2}\mathrm{e}^{\frac{-5g^2x^2}{U^4}}.$$

The expression for the free surface elevation in the physical space under this pressure distribution is then given by:

$$\eta_1(x) = \frac{U^2}{4g\sqrt{5\pi}}\,\mathrm{Re}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{k^2U^4}{20g^2}}e^{\mathrm{i}kx}}{|k| - \frac{4\mathrm{i}\nu k|k|}{U} - \frac{g}{U^2} + \frac{\varepsilon_p E_0^2|k|}{\rho U^2} - \frac{4\nu^2k^2|k|}{U^2} - \frac{\sigma k^2}{\rho U^2}}\,\mathrm{d}k. \tag{2.8}$$

A dispersion relation may be extracted from the above the by setting the denominator to zero, the dispersion relation is then:

$$U^2 - 4i\nu kU - \frac{g}{|k|} + \frac{\epsilon_p E_0^2}{\rho} - 4\nu^2k^2 - \frac{\sigma}{\rho}|k| = 0 \tag{2.9}$$

Expression (2.8) can be further simplified by choosing new dimensionless variables:

$$l := \frac{U^2}{g}k, \quad \hat{x} := \frac{g}{U^2}x, \quad \zeta := \frac{g}{U^2}\eta_1.$$

With these new variables the solution (2.8) becomes:

$$\zeta(\hat{x}) = \frac{1}{4\sqrt{5\pi}}\,\mathrm{Re}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{l^2}{20}}e^{\mathrm{i}l\hat{x}}}{(1+\beta)|l| - 1 - \mathrm{i}\delta l|l| - \gamma l^2|l| - \alpha l^2}\,\mathrm{d}l, \tag{2.10}$$

where

$$\alpha := \frac{\sigma g}{\rho U^4}, \quad \beta := \frac{\varepsilon_p E_0^2}{\rho U^2}, \quad \gamma := \frac{4\nu^2g^2}{U^6}, \quad \delta := \frac{4\nu g}{U^3}.$$

For instance, we can notice that the coefficient $\gamma = \mathcal{O}(\delta^2) \ll 1$ and, thus, can be neglected. The integrand in (2.10) is complex. The complex part plays the role of the Rayleigh dispersion as mentioned in [26]. From an analytical point of view, the complex terms remove the singularity. Consequently, formula (2.10) will always define a valid solution in L^2 due to the complex denominator. It is expected also that $\zeta(\hat{x}) \to 0$ as $\hat{x} \to \pm\infty$.

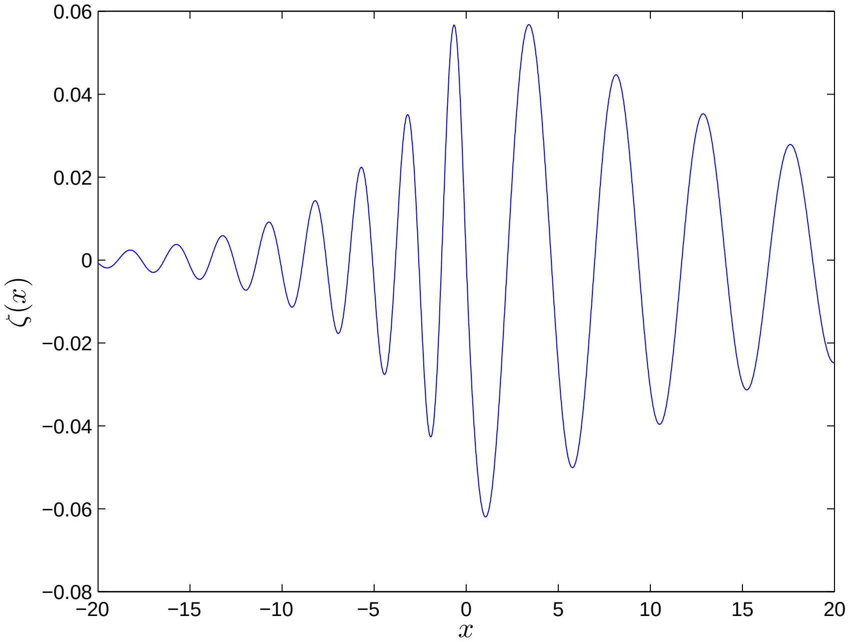

Figure 2. *Typical free surface profile predicted by solution* (2.10).

3. Results

Comparisons of (2.10) can be made with previous established results in the literature, taking $\beta = \gamma = \delta = 0$ reduces to the case in [26] and likewise setting $\gamma = \delta = 0$ reduces to the case in [11]. In order to illustrate the analytical result derived in the previous Section, we plot on Figure 2 the free surface elevation shape predicted analytically by solution (2.10). The parameters were chosen to be $\alpha = 0.3$, $\beta = 0.15$ and $\delta = 0.01$. The profile for these base values is given in Figure 2.

When $\nu = 0$, there will be two zeros in the denominator of (2.8) if parameters are taken so U is above the minimum of the dispersion relation (2.9). When this is the case Rayleigh dispersion is required for convergence of the integral. The weakly viscous terms acts as a kind of variable Rayleigh dispersion term which is why there is more damping than is normally seen with the inviscid case. In order to compare the viscous case with previous cases, the case in [11] is given by Figure 3.

As can be seen quite clearly, even a small viscosity has a large impact on wave of higher frequency by damping it severely and increases the amplitude of the lower frequency whilst also damping the waves. It is also interesting that the addition of viscosity doesn't affect the wavelength of the short and long waves. In order to illustrate the free surface elevation depends on the parameter β (which measures the relative importance of inertia to the electric force), we represent in Figure 4 the same solution (2.10) for several values of β The same result is plotted on Figure 5 also for the parameter δ, which measures the relative magnitude of dissipative effects. The increase in parameters leads the slight decrease of the amplitude, as it can be expected from the analytical solution.

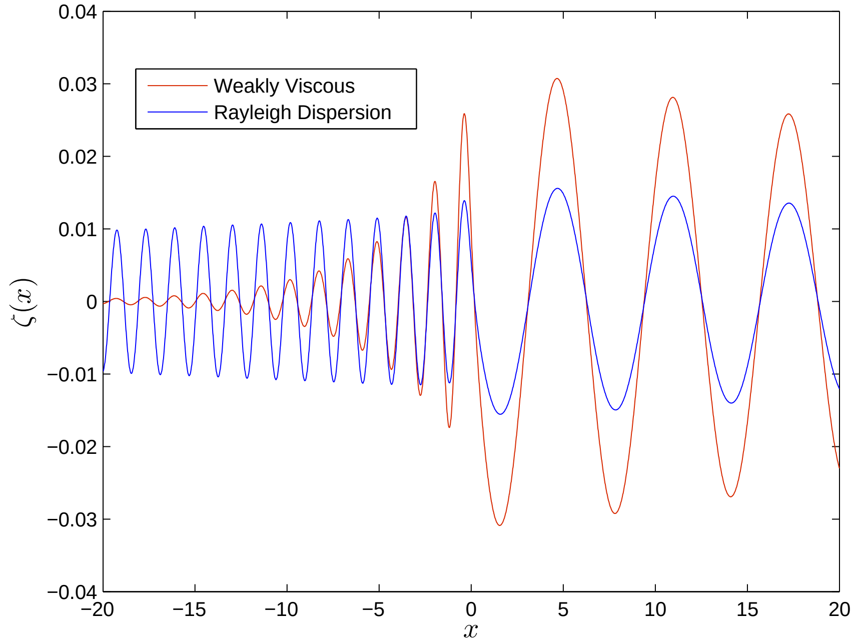

Figure 3. *Comparison of Non-Viscous Waves with Rayleigh Dispersion and Viscous Waves*

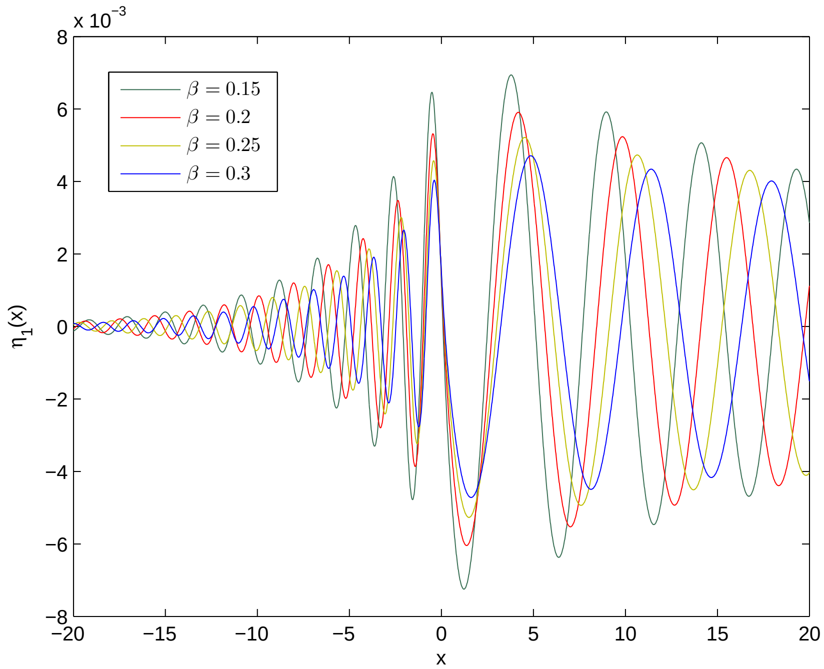

Figure 4. *Free Surface Profiles for Various Values of* β

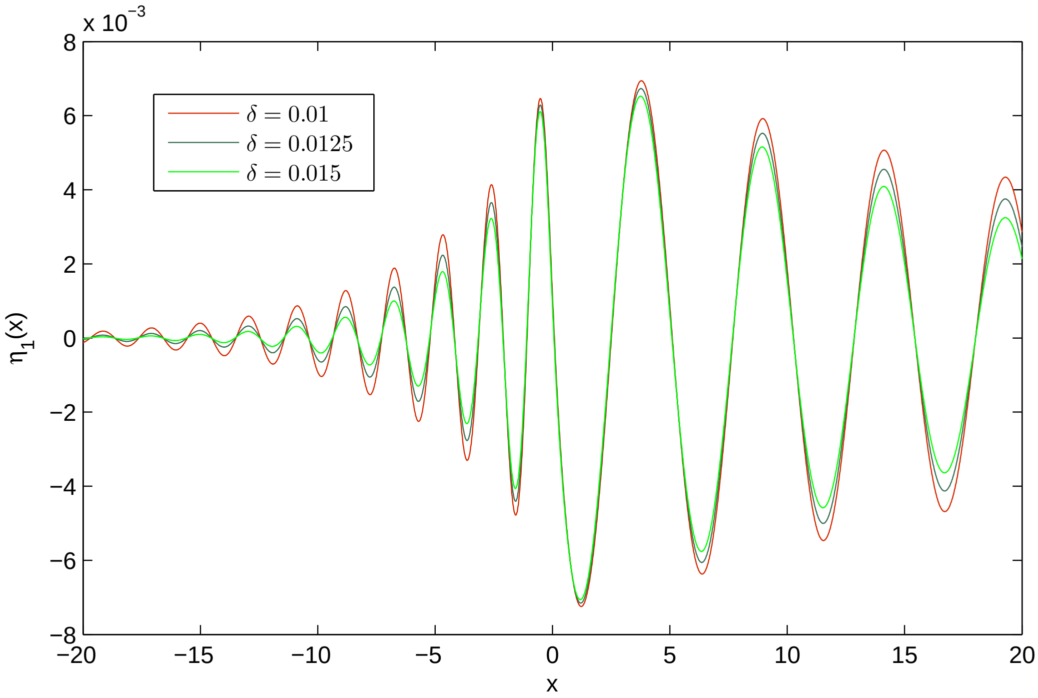

Figure 5. *Free surface elevation shapes for various values of the parameter δ.*

4. Conclusions & Perspectives

In this study we considered the problem of a free surface flow description in the presence of electric forces. Moreover, the fluid is assumed to be weakly viscous. By using the visco-potential flow theory, the classical potential formulation was modified to take into account weak dissipative effects. The derivations presented in this study were performed in the deep water approximation. However, the generalization to the finite depth case does not represent any major difficulties.

The free surface shape in the presence of a uniform current and a localized interfacial pressure distribution was computed analytically. The dependence of this shape on two important dimensionless parameters was studied numerically as well.

This study opens a certain number of directions for future investigations. For instance, the present analysis was only linear. Nonlinearities have to be taken into account as well as the finite depth effects. Moreover, long wave asymptotics can be also performed to investigate the shallow water regime.

Acknowledgments

D. Dutykh would like to acknowledge the support from ERC under the research project ERC-2011-AdG 290562-MULTIWAVE. The authors would like to thank the anonymous referees whose suggestions have greatly improved this article.

A. Electric charge at the interface

This section details the analysis of the problem where there is an interfacial surface charge. In the inviscid case, as there were co shearing forces, there could be no dynamic charge on the surface and it was only possible to compute in *induced* charge on the interface given by:

$$\Sigma_Q = \epsilon_p \frac{\partial V}{\partial n}$$

Then the definition leads to the same result as in the inviscid case, that the induced surface charge is just the Hilbert transform of the derivative of the free surface profile. For the *dynamics* surface charge equation (2.6) is replaced with two following equations [10]:

$$\hat{\mathbf{n}} \cdot \left[\varepsilon_p \mathbf{E}\right]_1^2 = q,$$

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla}_T \cdot (q\mathbf{u}) + \hat{\mathbf{n}}\left[\sigma_q \mathbf{E}\right]_1^2 = 0.$$

The operator $\boldsymbol{\nabla}_T$ is the covariant derivative defined as

$$\boldsymbol{\nabla}_T = \boldsymbol{\nabla} - \hat{\mathbf{n}}\big(\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\big).$$

Below we will perform the linear analysis of this problem as well. The expansions in this case are given by:

$$\begin{aligned}
\varphi &= Ux + \varepsilon\varphi_1 + o(\varepsilon) \\
\eta &= \varepsilon\eta_1 + o(\varepsilon) \\
V &= -E_0 x + \varepsilon V_1 + o(\varepsilon) \\
q &= \varepsilon q_1 + o(\varepsilon) \\
p &= \varepsilon p_1
\end{aligned}$$

The covariant derivative on the free surface with the normal defined by equation (2.4) is expressed as

$$\boldsymbol{\nabla}_T = \Big(\frac{\partial}{\partial x}, 0\Big) + \varepsilon\Big(\frac{\partial \eta_1}{\partial x}\frac{\partial}{\partial y}, \frac{\partial \eta_1}{\partial x}\frac{\partial}{\partial x}\Big) + o(\varepsilon)$$

The equations reduce to the following [10]:

$$U\frac{\partial q_1}{\partial x} + \sigma_q E_0 \frac{\partial \eta_1}{\partial x} + \sigma_q \frac{\partial V_1}{\partial y} = 0$$

$$-\varepsilon_p E_0 \frac{\partial \eta_1}{\partial x} + \varepsilon_p \frac{\partial V_1}{\partial y} = q_1$$

Two last equations can be combined into one:

$$-\varepsilon_p E_0 U \frac{\partial^2 \eta_1}{\partial x^2} + U\varepsilon_p \frac{\partial^2 V_1}{\partial x \partial y} + \sigma_q E_0 \frac{\partial \eta_1}{\partial x} + \sigma_q \frac{\partial V_1}{\partial y} = 0.$$

The rest of linearised equations in this case is exactly the same as in the previous section. The analytical expression for the linearized free surface elevation in the Fourier space can be derived in a similar way:

$$\hat{\eta}_1(k) = \left[k\,\mathrm{sgn}(k) - \frac{4\mathrm{i}\nu k^2\,\mathrm{sgn}(k)}{U} - \frac{g}{U^2} + \frac{\varepsilon_p E_0^2}{\rho U^2} k f(k) - \frac{\sigma}{\rho U^2}k^2\right]^{-1} \frac{\hat{p}_1(k)}{\rho U^2}$$

Then, the free surface elevation in the physical space can be easily obtained:

$$\eta_1(x) = \frac{1}{2\pi\rho U^2}\int_{\mathbb{R}} \frac{\hat{p}_1(k)\mathrm{e}^{\mathrm{i}kx}}{k\,\mathrm{sgn}(k) - \frac{4\mathrm{i}\nu k^2\,\mathrm{sgn}(k)}{U} - \frac{g}{U^2} + \frac{\varepsilon_p E_0^2}{\rho U^2}kf(k) - \frac{\sigma}{\rho U^2}k^2}\,\mathrm{d}k, \quad f(k) = \frac{\varepsilon_p Uk + \mathrm{i}\sigma_q}{\sigma_q + \mathrm{i}kU\varepsilon_p}$$

Using the same scalings and pressure form as before, the integral can be reduced to the following dimensionless form:

$$\zeta(\hat{x}) = \frac{1}{4\sqrt{5\pi}}\,\mathrm{Re}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\frac{l^2}{20}}\mathrm{e}^{\mathrm{i}\hat{x}l}}{|l| - \mathrm{i}\delta l|l| - 1 + \beta l\tilde{f}(l) - \alpha l^2}\,\mathrm{d}l,$$

where

$$\tilde{f} = \frac{\mu l + \mathrm{i}}{\mathrm{i}\mu l + 1}, \qquad \mu = \frac{\varepsilon_p g}{U\sigma_q},$$

and coefficients α, β and δ are defined as in the previous section.

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