Cubic Equations

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1 Derivation of Solution

A cubic equation is an algebraic equation of the form:

$$Ax^3 + Bx^2 + Cx + D = 0 \tag{1}$$

It will be assumed that $A \leq 0$, otherwise the equation will be a quadratic, and the solution of these equations are well known and not will concern us here. Divide by A to obtain the equation:

$$x^3 + \frac{B}{A}x^2 + \frac{C}{A}x + \frac{D}{A} = 0. \tag{2}$$

Write $A' = B/A$, $B' = C/A$ and $C' = D/A$ to get the equation

$$x^3 + A'x^2 + B'x + C' = 0 \tag{3}$$

Make the substitution:

$$x = y - \frac{A'}{3} \tag{4}$$

Then:

$$\begin{aligned}
x^3 &= y^3 - A'y^2 + A'^2y - \frac{A'^3}{27} \\
X^2 &= y^2 - \frac{2A'y}{3} + \frac{A'^2}{9} \\
x &= y - \frac{A'}{3}
\end{aligned}$$

So inserting these into (3) shows that:

$$\begin{aligned}
0 &= y^3 - A'y^2 + A'^2y - \frac{A'^3}{27} + A'\left(y^2 - \frac{2A'y}{3} + \frac{A'^2}{9}\right) + \\
&\quad + B'\left(y - \frac{A'}{3}\right) + C' \\
&= y^3 + \left(A'^2 - \frac{2A'^2}{3} + B'\right)y + \left(\frac{2A'^3}{27} - \frac{A'B'}{3} + C'\right)
\end{aligned}$$

So this equation can be written as:

$$y^3 + ay + b = 0 \tag{5}$$

where:

$$a = A'^2 - \frac{2A'^2}{3} + B', \quad b = \frac{2A'^3}{27} - \frac{A'B'}{3} + C' \tag{6}$$

The next step seems rather trivial, let $y = u + v$, then:

$$\begin{aligned} y^3 &= (u+v)^3 \\ &= u^3 + 3u^2v + 3uv^2 + v^3 \\ &= u^3 + v^3 + 3uv(u+v) \\ &= u^3 + v^3 + 3uvy \\ \Rightarrow y^3 - 3uvy - u^3 - v^3 &= 0 \end{aligned}$$

Then:

$$a = -3uv, \quad b = -u^3 - v^3 \tag{7}$$

Write $v = -s/3v$ and substitute it into the second equation of (7) to obtain:

$$b = -u^3 - \left(-\frac{a}{3u}\right)^3 \Rightarrow b = -u^3 + \frac{a^3}{27u^3} \Rightarrow u^6 + bu^3 - \frac{a^3}{27} = 0 \tag{8}$$

This is a quadratic equation in u^3, the solution can be just written down:

$$u^3 = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \tag{9}$$

Now if we wrote $u = -1/3v$ and substitute it into the second equation of equation (7) we obtain the same quadratic as we did in (8). So we can write:

$$u = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}, \quad v = \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} \tag{10}$$

So adding these two solutions gives the solution y

$$y = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} \tag{11}$$

So the solution for x is given by:

$$\boxed{x = -\frac{B}{3A} + \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}} \tag{12}$$

2 Example

Suppose we have to solve the cubic:

$$2x^3 - 7x^2 + 7x - 2 = 0 \tag{13}$$

So we go through our process, divide through by the coefficient of x^3 to obtain:

$$x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0 \tag{14}$$

Now make the substitution of $x = y + 7/6$, and noting that:

$$\begin{aligned} x^3 &= y^3 + \frac{7}{2}y^2 + \frac{7^2}{12}y + \frac{7^3}{6^3} \\ x^2 &= y^2 + \frac{7}{3}y + \frac{7^2}{6^2} \end{aligned}$$

Inserting this into equation (14) shows that:

$$\begin{aligned} 0 &= y^3 + \frac{7}{2}y^2 + \frac{7^2}{12}y + \frac{7^3}{6^3} - \frac{7}{2}\left(y^2 + \frac{7}{3}y + \frac{7^2}{6^2}\right) + \\ &\quad + \frac{7}{2}\left(y + \frac{7}{6}\right) - 1 \\ &= y^3 + \left(\frac{7^2}{12} - \frac{7^2}{6} + \frac{7}{2}\right)y + \left(\frac{7^3}{6^3} - \frac{7^3}{2 \times 6^2} + \frac{7^2}{12} - 1\right) \end{aligned}$$

Which reduces to:

$$y^3 - \frac{7}{12}y - \frac{5}{54} = 0 \tag{15}$$

So:

$$a = -\frac{7}{12}, \quad b = -\frac{5}{54} \tag{16}$$

We note that:

$$\sqrt{\frac{b^2}{4} + \frac{a^3}{27}} = 0.072169i \tag{17}$$

Then we also compute:

$$-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} = 0.046296 + 0.072169i \tag{18}$$

We are required to compute the cube root of this. To do this, we write this in exponential form:

$$0.046296 + 0.072169i = re^{i\theta} \tag{19}$$

To compute r, we calculate $r = \sqrt{0.046296^2 + 0.072169^2} = 0.085742$. Then to compute the angle, θ, $\theta = \tan^{-1}(0.072169/0.046296) = 1.0004$ Then the complex number becomes:

$$0.046296 + 0.072169i = 0.085742e^{1.0004i} \tag{20}$$

We are then required to take the cube root of this complex number, to do this we use De Moivre's theorem $(re^{i\theta})^k = r^k e^{ik\theta}$ to compute the cube root:

$$\left(0.085742e^{1.0004i}\right)^{\frac{1}{3}} = 0.44096e^{0.33347i} = 0.41667 + 0.14434i \tag{21}$$

Using the same process (a useful exercise) we can write:

$$\left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} = 0.41667 - 0.14434i \tag{22}$$

Then finally we can compute x:

$$x = \frac{7}{6} + 0.41667 + 0.14434i + 0.41667 - 0.14434i = 2 \tag{23}$$

Then we can write:

$$2x^3 - 7x^2 + 7x - 2 = (x-2)(ax^2 + bx + c) \tag{24}$$

Expanding shows that:

$$ax^3 + (b-2a)x^2 + (x-2b)x - 2c = 2x^3 - 7x^2 + 7x - 2 \tag{25}$$

A comparison of coefficients shows that $a = 2$, $b = -3$ and hence:

$$2x^3 - 7x^2 + 7x - 2 = (x-2)(2x^2 - 3x + 1) \tag{26}$$

We know how to solve quadratics and it's simple to factorise:

$$2x^2 - 3x + 1 = (2x-1)(x-1) \tag{27}$$

So finally we have solved our cubic:

$$2x^3 - 7x^2 + 7x - 2 = (2x-1)(x-1)(x-2) \tag{28}$$